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Topological defects for the free boson CFT

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Abstract
Two different conformal field theories can be joined together along a defect line. We study such defects for the case where the conformal field theories on either side are single free bosons compactified on a circle. We concentrate on topological defects for which the left- and right-moving Virasoro algebras are separately preserved, but not necessarily any additional symmetries. For the case where both radii are rational multiples of the self-dual radius we classify these topological defects. We also show that the isomorphism between two T-dual free boson conformal field theories can be described by the action of a topological defect, and hence that T-duality can be understood as a special type of order–disorder duality.

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Contents

1. Introduction

In this paper, we investigate how one can join two free boson conformal field theories along a line in a conformally invariant manner. More specifically, we are interested in interfaces which preserve the left- and right-moving conformal symmetries separately. Such interfaces are special types of conformal defects that appear naturally in conformal field theory; conformal defects have recently attracted some attention, see e.g. [1–12]. As we shall review momentarily, the special (topological) defects we consider in this paper have a number of interesting and useful properties.

Let us consider the case that the world sheet is the complex plane and the interface runs along the real axis. We take the conformal field theory on the upper half plane to be the compactified free boson of radius R_1 and the theory on the lower half plane to be the free boson theory at radius R_2. Denoting the left- and right-moving stress tensors of the two theories by T^i and $\bar{T}^i$, $i = 1, 2$, respectively, general conformal defects are interfaces that obey

$$T^1(x) - \bar{T}^1(x) = T^2(x) - \bar{T}^2(x) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{1.1}$$

Via the folding trick [13], solutions to (1.1) correspond to conformal boundary conditions for the product theory consisting of two free bosons compactified to R_1 and R_2, respectively. (From a target space perspective we are thus looking for the conformal D-branes of the theory on a rectangular torus with radii R_1 and R_2.) This method to investigate the conformal interfaces was used in [5].

While all conformal boundary conditions for the theory of a single free boson on a circle are known [14–17], the classification of all conformal boundaries of the $c = 2$ theory in question is presently out of reach. Thus we cannot hope to find all interfaces obeying (1.1).

There is, however, an interesting subclass of conformal interfaces for which a classification can be achieved. These are the interfaces for which the condition (1.1) is strengthened to

$$T^1(x) = T^2(x) \qquad \text{and} \qquad \bar{T}^1(x) = \bar{T}^2(x) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{1.2}$$

In other words, we require that the stress tensor is continuous across the interface. In this case, the interface commutes with the generators of local conformal transformations, and the interface line can be continuously deformed without affecting the value of correlators as long it does not cross any field insertion points. These interfaces are therefore called *topological defects* [10] (or sometimes also totally transmissive defects). They have, at least, two nice properties:

- Topological defects can be fused together (by letting the interfaces merge), and thus carry a multiplicative structure [1–4, 11]. They therefore possess more structure than the corresponding boundary conditions.
- Topological defects contain information about symmetries of the conformal field theory (in the present case the free boson), as well as about order–disorder dualities [9, 11]. For the free boson we identify a topological defect of the latter type, which generates the T-duality symmetry.

In this paper, we shall describe a large class of such topological defects for the compactified free boson and give a classification for specific values of the radii. We should stress that even if the two radii are different, $R_1 \neq R_2$, many topological defects exist—this will become clear in section 4; the precise form of the defects then depends on the arithmetic properties of R_1 and R_2. Furthermore, via the folding trick, these topological defects correspond to new conformal boundary conditions in the product of two free boson CFTs.

It is difficult to verify that a given collection of topological defects is consistent. In principle one has to specify all correlators involving such defect lines and verify the relevant sewing conditions. (A complete list of sewing constraints for correlators involving defects has not been written down; it will be an extension of the constraints for correlators involving boundaries given in [18, 19].). In this paper, we take two approaches to this problem. The first is to use the TFT formulation for constructing CFT correlators [7, 11, 20], which does give all collections of correlators (involving boundaries and topological defect lines) that are consistent with sewing. However, this approach only applies to *rational* conformal field theories and to the defects which preserve a rational symmetry. This approach thus gives a collection of defects that are guaranteed to be consistent, but these will not be all. We can, however, give a criterion for a deformation of topological defects by a defect field to be exactly marginal, and in this way we will also gain some information about the neighbourhood of these 'rational' topological defects.

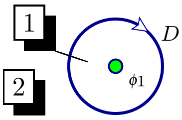
The second approach is to select some of the necessary consistency conditions which are relatively easy to analyse (at least for the free boson) and to try to classify all of their solutions. One example is the analogue of the Cardy constraint for boundary conditions [21]; it arises from analysing torus partition functions with insertions of defect lines [1]. We will also use an additional condition which is obtained by deforming defect lines in the presence of bulk fields. This gives an upper bound on the complete list of consistent defects. By comparing the results from the two approaches we can thus propose a fairly convincing picture of what all the consistent defects between free boson theories at $c = 1$ are.

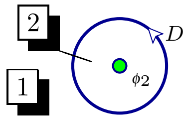
The paper is organized as follows. In section 2, we give a brief introduction to topological defects, section 3 contains our conventions for the free boson theory and in section 4 we give a non-technical summary of our results. The details of the first approach are presented in section 5 and those of the second approach in section 6. Some technical calculations have been collected in several appendices.

2. Topological defect lines

A defect is a one-dimensional interface separating two conformal field theories CFT_1 and CFT_2. The defect is called conformal iff the stress–energy tensors of the two theories are related as in (1.1). In the following, we shall concentrate on topological defects for which the stronger condition $T^1 = T^2$ and $\overline{T}^1 = \overline{T}^2$ holds, see (1.2). In other words, we shall require that the two independent components of the stress–energy tensor are continuous across the defect line.

Suppose D is a topological defect joining the two conformal field theories CFT_1 and CFT_2, and denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ the corresponding spaces of bulk states. The topological defect D then gives rise to a linear operator $\hat{D} : \mathcal{H}_1 \to \mathcal{H}_2$ which is obtained by taking D to wind around the field insertion with the appropriate orientation[5]. The defect $\bar{D}$ with the opposite orientation then defines a map $\hat{\bar{D}} : \mathcal{H}_2 \to \mathcal{H}_1$. Pictorially,



$=: (\hat{D}\phi_1)(z)$ and


$=: (\hat{\bar{D}}\phi_2)(z).$ (2.1)

As will be discussed in section 4.5, these operators do not specify the defect uniquely. They do, however, contain important information about the defect. For example, just like boundary conditions, defects also possess excitations that are localized on the defect; these are called defect fields. The spectrum of defect fields can be determined by considering the trace of $\hat{\bar{D}}\hat{D}$ in $\mathcal{H}_1$. Furthermore, defect operators determine the correlators of two bulk fields on a sphere which are separated by the defect loop. These correlators play an analogous role for topological defects as the one-point correlators of bulk fields on the disc do for conformal boundary conditions. Defect operators finally provide the point of view from which topological defects were first studied in [1].

Since a topological defect D is transparent to T and $\bar{T}$, the corresponding operator commutes with the Virasoro modes, i.e.,

$$L^2_m \hat{D} = \hat{D} L^1_m \qquad \text{and} \qquad \bar{L}^2_m \hat{D} = \hat{D} \bar{L}^1_m, \tag{2.2}$$

where L^i_m and $\bar{L}^i_m$ act on the state space $\mathcal{H}_i$. A similar statement also holds for $\hat{\bar{D}}$. However, not every map $\hat{D}$ satisfying (2.2) arises as the operator of a topological defect. This is analogous to the case of boundary conditions: (2.2) corresponds to the conformal gluing condition, but not every solution of the conformal gluing condition defines a consistent boundary state—for example, the boundary states must in addition satisfy the Cardy condition, etc.

Topological defects can be fused. We denote the fusion of two defects D_1 and D_2 by $D_1 * D_2$. In terms of the associated operators $\hat{D}$, fusion just corresponds to the composition $\hat{D}_1 \circ \hat{D}_2$. Note that in general the fusion of defects is not commutative. Let D be a defect between two copies of the same conformal field theory, i.e. $\mathrm{CFT}_1 = \mathrm{CFT}_2$. Then we call D *group-like* iff fusing it with the defect of opposite orientation yields the trivial defect, $\bar{D} * D = \mathbf{1}$. (The trivial defect between two identical conformal field theories corresponds simply to the identity map.) The group-like defects form a group. It can be shown that this group describes internal symmetries for CFT correlators on world sheets of arbitrary genus [11].

We can also form superpositions of topological defects, which corresponds to adding the associated operators. We call a defect operator $\hat{D}$ *fundamental* iff it is nonzero and it cannot be

[5] Here, as in the following, we shall identify bulk fields $\phi(z)$ with the corresponding states ϕ in $\mathcal{H}$.

written as a sum of two other nonzero defect operators. Finally, a topological defect D is called a *duality defect* iff $\bar{D} * D$ decomposes into a superposition of only group-like defects (thus group-like defects are a special case of duality defects). Duality defects relate correlators of the theories CFT_1 and CFT_2 and describe order–disorder, or Kramers–Wannier-type, dualities [9, 11].

As we will see, T-duality is generated by a special kind of duality defect. In general, duality defects give rise to identities that relate correlators of bulk fields to correlators of disorder fields. (Disorder fields are those defect fields that appear at the end points of defect lines.) In the case of T-duality, the resulting disorder fields are in fact again local bulk fields. Thus, T-duality can be understood as a special type of order–disorder duality, an observation also made in [22, 23] on the basis of lattice discretizations.

3. The compactified free boson

Before discussing topological defects in detail, let us fix our conventions for the free boson compactified on a circle of radius R. The chiral symmetry of the free boson $\phi(z)$ is a $u(1)$ current algebra generated by $J(z) = \mathrm{i}\sqrt{\frac{2}{\alpha'}}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z)$ with operator product expansion

$$J(z)J(0) = z^{-2} + \mathrm{reg}. \tag{3.1}$$

Its irreducible highest weight representations $\mathcal{H}_q$ are uniquely characterized by the eigenvalue of the zero mode J_0 on the highest weight state $|q\rangle$, $J_0|q\rangle = q|q\rangle$. The stress–energy tensor is $T(z) = \frac{1}{2}{:}J(z)J(z){:}$, so that the conformal weight of $|q\rangle$ is $h = \frac{1}{2}q^2$.

For the boson of radius R, the bulk spectrum can be written as a direct sum

$$\mathcal{H}(R) = \bigoplus_{m,w\in\mathbb{Z}} \mathcal{H}_{q_{m,w}(R)} \otimes \bar{\mathcal{H}}_{\bar{q}_{m,w}(R)} \tag{3.2}$$

of representations of the left- and right-moving current algebras, where

$$q_{m,w}(R) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R}m + \frac{R}{\sqrt{\alpha'}}w\right), \qquad \bar{q}_{m,w}(R) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R}m - \frac{R}{\sqrt{\alpha'}}w\right). \tag{3.3}$$

The integer w is the winding number and m is related to the total momentum p via $p = (2\alpha')^{-1/2}(q_{m,w} + \bar{q}_{m,w}) = m/R$. The charges $(q, \bar{q})$ that appear in the decomposition (3.2) of $\mathcal{H}(R)$ form the *charge lattice* $\Lambda(R)$. It is generated by the two vectors

$$\frac{\sqrt{\alpha'}}{\sqrt{2}R}(1, 1) \qquad \text{and} \qquad \frac{R}{\sqrt{2\alpha'}}(1, -1) \tag{3.4}$$

which describe a pure momentum and pure winding state, respectively.

The chiral vertex operator corresponding to the highest weight state $|q\rangle$ is given by the normal ordered exponential ${:}\mathrm{e}^{\mathrm{i}\sqrt{2/\alpha'}q\phi(z)}{:}$. The bulk field corresponding to the highest weight state in the sector $(q, \bar{q}) \in \Lambda(R)$ is obtained from the product of two such normal ordered exponentials; it will be denoted by $\phi_{(q,\bar{q})}$. The bulk fields can be normalized in such a way that the operator products take the form

$$\phi_{(q_1,\bar{q}_1)}(z)\phi_{(q_2,\bar{q}_2)}(w) = (-1)^{m_1 w_2}(z - w)^{q_1 q_2}(z^* - w^*)^{\bar{q}_1\bar{q}_2}\big(\phi_{(q_1+q_2,\bar{q}_1+\bar{q}_2)}(w) + O(|z - w|)\big). \tag{3.5}$$

The factor $(-1)^{m_1 w_2}$ is needed for locality; in particular, one cannot set all OPE coefficients to 1.[6]

The conformal field theory of a free boson compactified at radius R as described above will be denoted by $\mathrm{Bos}(R)$. It is not difficult to see that the bulk spectrum (3.2) is invariant under the substitution $R \mapsto \alpha'/R$; this is the usual T-duality relation for the compactified free boson.

If applied to the perturbative expansion of the string free energy $F(R, g_s)$, one must take into account that T-duality also acts on the dilaton field (see, e.g., [24]). As a result, at the same time as changing the radius one must also modify the string coupling constant, $F(R, g_s) = F(R', g_s')$, with

$$R' = \frac{\alpha'}{R}, \qquad g_s' = \frac{\sqrt{\alpha'}}{R} g_s. \tag{3.6}$$

We will recover this change in g_s when analysing the description of T-duality in terms of topological defects in section 5.4.

In the following, we shall set the value of the parameter α' to

$$\alpha' = \tfrac{1}{2}. \tag{3.7}$$

With this choice, the radius T-dual to R is $R' = 1/(2R)$, and the self-dual radius is

$$R_{\mathrm{s.d.}} = 1/\sqrt{2}. \tag{3.8}$$

To restore the α'-dependence in the expressions below, one simply has to substitute all appearances of R by $R/\sqrt{2\alpha'}$.

4. Topological defects for the free boson

Before going through details of the calculations, let us explain and summarize the results found in sections 5 and 6. We will explicitly give operators $\hat{D}$ of the various topological defects and compute their compositions.

4.1. Topological defects preserving the $\widehat{u}(1)$-symmetry

The simplest topological defects are those that actually preserve more symmetry than just the two Virasoro symmetries (2.2). In particular, we can demand that the topological defect also intertwines the $\widehat{u}(1)$-symmetries up to automorphisms,

$$J^2_m \hat{D} = \epsilon \hat{D} J^1_m \qquad \text{and} \qquad \bar{J}^2_m \hat{D} = \bar{\epsilon} \hat{D} \bar{J}^1_m \qquad \text{for} \quad \epsilon, \bar{\epsilon} \in \{\pm 1\}. \tag{4.1}$$

This condition implies (2.2). From the action of the zero modes $J_0^{1,2}$ and $\bar{J}_0^{1,2}$ we see that $\hat{D}$ has to map the highest weight state $(q, \bar{q}) \in \Lambda(R_1)$ to the highest weight state $(\epsilon q, \bar{\epsilon}\bar{q}) \in \Lambda(R_2)$. Thus $\hat{D}$ can only be nonzero in sectors for which $(q, \bar{q})$ lies in the intersection Λ of the two lattices,

$$\Lambda = \Lambda^{\epsilon,\bar{\epsilon}}(R_2) \cap \Lambda(R_1), \qquad \text{where} \quad \Lambda^{\epsilon,\bar{\epsilon}}(R) = \{(\epsilon q, \bar{\epsilon}\bar{q}) | (q, \bar{q}) \in \Lambda(R)\}. \tag{4.2}$$

[6] Consider, for example, two fields ϕ_1 and ϕ_2 with $(m_1, w_1) = (1, 0)$ and $(m_2, w_2) = (0, 1)$. The operator products $\phi_1(z)\phi_2(w)$ and $\phi_2(z)\phi_1(w)$ must be related by analytic continuation. Since in both cases the leading singularity is $(z - w)^{\frac{1}{2}}(z^* - w^*)^{-\frac{1}{2}}$, the sign arising in the analytic continuation must be compensated by the OPE coefficient. Note also that in the convention (3.5) some two-point functions are negative. This can be avoided at the cost of introducing imaginary OPE coefficients. Namely, in terms of the basis $\phi'_{(q,\bar{q})} = \mathrm{i}^{mw}\phi_{(q,\bar{q})} = \mathrm{e}^{\mathrm{i}\pi(q^2-\bar{q}^2)/4}\phi_{(q,\bar{q})}$ the OPE coefficients are $\mathrm{i}^{m_1 w_2 - m_2 w_1}$.

To describe this lattice more explicitly, we observe that

$$\Lambda^{\epsilon,\bar\epsilon}(R) = \Lambda(\hat R), \qquad \text{where} \quad \hat R := \begin{cases} R & \text{if} \quad \epsilon = \bar\epsilon, \\ \frac{1}{2R} & \text{if} \quad \epsilon = -\bar\epsilon, \end{cases} \tag{4.3}$$

as follows directly from (3.4). Thus, the intersection Λ consists of all points such that

$$\frac{m}{2R_1}\begin{pmatrix}1\\1\end{pmatrix} + wR_1\begin{pmatrix}1\\-1\end{pmatrix} = \frac{m'}{2\hat R_2}\begin{pmatrix}1\\1\end{pmatrix} + w'\hat R_2\begin{pmatrix}1\\-1\end{pmatrix} \qquad \text{for some} \quad m, w, m', w' \in \mathbb{Z}. \tag{4.4}$$

In particular, this means that $m = (R_1/\hat R_2)m'$ and $w = (\hat R_2/R_1)w'$. We will treat separately the cases that $\hat R_2/R_1$ is rational or irrational.

4.1.1. $\hat R_2/R_1$ rational. Let us write $\hat R_2/R_1 = M/N$, where M and N are coprime positive integers. It then follows from the discussion above that the lattice Λ is spanned by the vectors $N/(2R_1)\cdot(1,1)$ and $MR_1\cdot(1,-1)$. Because of (4.1) the defect operator is fixed on each sector $\mathcal{H}_q \otimes \bar{\mathcal{H}}_{\bar q}$ of $\mathcal{H}(R_1)$ once we know its action on the primary bulk fields $\phi_{(q,\bar q)}$ chosen in section 3. One finds that the possible defect operators are parametrized by two complex numbers x and y and that they act on the primary bulk fields as

$$\hat D(x,y)^{\epsilon,\bar\epsilon}_{R_2,R_1}\phi_{(q,\bar q)} = \sqrt{MN}(\epsilon\bar\epsilon)^{\frac12 q^2-\frac12\bar q^2}\,\mathrm{e}^{2\pi\mathrm{i}(xq-y\bar q)}\phi_{(\epsilon q,\bar\epsilon\bar q)} \tag{4.5}$$

if $(q,\bar q)\in\Lambda$ and as zero otherwise. Note that the exponent of the sign $\epsilon\bar\epsilon$ is an integer, since it is the difference $h-\bar h$ of the left and right conformal weights of the $\widehat{u}(1)$-primary field with charge $(q,\bar q)$. The complete defect operator can be written as

$$\hat D(x,y)^{\epsilon,\bar\epsilon}_{R_2,R_1} = \sqrt{MN}\sum_{(q,\bar q)\in\Lambda}(\epsilon\bar\epsilon)^{\frac12 q^2-\frac12\bar q^2}\,\mathrm{e}^{2\pi\mathrm{i}(xq-y\bar q)}\,P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q}, \tag{4.6}$$

where $P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q} : \mathcal{H}(R_1)\to\mathcal{H}(R_2)$ is the twisted intertwiner uniquely determined by

$$\begin{aligned} &P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q}\phi_{(q',\bar q')} = \delta_{q,q'}\delta_{\bar q,\bar q'}\phi_{(\epsilon q,\bar\epsilon\bar q)} \\ &\text{and} \qquad J_m P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q} = \epsilon P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q} J_m, \qquad \bar J_m P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q} = \bar\epsilon P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q}\bar J_m. \end{aligned} \tag{4.7}$$

As will be explained in section 6.2, the specific scalar coefficients that multiply the maps $P^{\epsilon q,\bar\epsilon\bar q}_{q,\bar q}$ for each sector in Λ can be determined by requiring consistency with the bulk OPE (3.5). One can also verify that the defects (4.6) give consistent torus amplitudes, i.e. integer multiplicities in the channel in which the defects run parallel to the Euclidean time direction. Furthermore, when R_1 and R_2 square to rational numbers, some of the defects (4.6) can be analysed from the point of view of the extended chiral symmetry. This is done in section 5; it demonstrates that at least these defects are consistent with all other sewing conditions as well. This leads us to believe that in fact all the operators (4.6) come from consistent defects.

In (4.6) x and y are *a priori* arbitrary *complex* constants. However, unless $x, y\in\mathbb{R}$ the spectrum of defect-changing fields may contain complex conformal weights. (This is similar to the situation in [15].) Furthermore, not all values of x and y lead to distinct defect operators; in fact, it follows directly from (4.6) that

$$\hat D(x,y)^{\epsilon,\bar\epsilon}_{R_2,R_1} = \hat D(x',y')^{\epsilon',\bar\epsilon'}_{R_2,R_1} \qquad \text{iff} \quad \epsilon=\epsilon', \bar\epsilon=\bar\epsilon', (x-x', y-y')\in\Lambda^*, \tag{4.8}$$

where Λ^* is the lattice dual to Λ, consisting of all points $(x,y)\in\mathbb{R}^2$ such that $xq-y\bar q\in\mathbb{Z}$ for all $(q,\bar q)\in\Lambda$.

It is also straightforward to read off the composition rules by rewriting the composition of two defect operators as a sum over operators of the form (4.6). One finds, for example,

$$\begin{aligned}
&\hat{D}(x,y)^{\epsilon,\epsilon}_{R,R} \circ \hat{D}(u,v)^{\nu,\nu}_{R,R} = \hat{D}(\nu x+u, \nu y+v)^{\epsilon\nu,\epsilon\nu}_{R,R},\\
&\hat{D}(0,0)^{\epsilon,\bar\epsilon}_{R_2,R_1} \circ \hat{D}(x,y)^{+,+}_{R_1,R_1} = \hat{D}(x,y)^{\epsilon,\bar\epsilon}_{R_2,R_1} = \hat{D}(\epsilon x, \bar\epsilon y)^{+,+}_{R_2,R_2} \circ \hat{D}(0,0)^{\epsilon,\bar\epsilon}_{R_2,R_1},\\
&\hat{D}(0,0)^{\epsilon,\bar\epsilon}_{R_1,R_2} \circ \hat{D}(0,0)^{\epsilon,\bar\epsilon}_{R_2,R_1} = \sum_{m=0}^{M-1}\sum_{w=0}^{N-1} \hat{D}\left(\frac{m}{2MR_1}+\frac{wR_1}{N}, \frac{m}{2MR_1}-\frac{wR_1}{N}\right)^{+,+}_{R_1,R_1}.
\end{aligned} \tag{4.9}$$

The equalities in the second and third lines imply that a defect with operator $\hat{D}(x,y)^{\epsilon,\bar\epsilon}_{R,R}$ is group-like if and only if $M = N = 1$. This is the case either if $\epsilon = \bar\epsilon$ and R is arbitrary (in this case our results have also been confirmed using a geometric realization of the defect lines [25]) or if $\epsilon = -\bar\epsilon$ and R is the self-dual radius. The group $\mathcal{G}^{u(1)}_R$ of $\widehat{u}(1)$-preserving group-like defect operators for $\mathrm{Bos}(R)$ is thus

$$\mathcal{G}^{u(1)}_R = \begin{cases} (\mathbb{C}^2/\Lambda(R)) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2) & \text{if} \quad R = \frac{1}{\sqrt{2}},\\ (\mathbb{C}^2/\Lambda(R)) \rtimes \mathbb{Z}_2 & \text{else,} \end{cases} \tag{4.10}$$

where we use that $\Lambda(R)^* = \Lambda(R)$ and abbreviate $\mathbb{Z}/(a\mathbb{Z})$ by $\mathbb{Z}_a$. The multiplication rule in the two cases is given by

$$\begin{aligned}
&(x,y,\epsilon,\bar\epsilon)\cdot(u,v,\nu,\bar\nu) = (\nu x+u, \bar\nu y+v, \epsilon\nu, \bar\epsilon\bar\nu) && \text{if} \quad R = \tfrac{1}{\sqrt{2}},\\
&(x,y,\epsilon)\cdot(u,v,\nu) = (\nu x+u, \nu y+v, \epsilon\nu) && \text{else,}
\end{aligned} \tag{4.11}$$

so that $\mathbb{Z}_2$'s are realized multiplicatively as $\{\pm 1\}$. Note that the group (4.10) is non-Abelian in all cases.

Furthermore, we see from (4.9) that *all* $\widehat{u}(1)$-preserving defects are duality defects. The duality defect implementing T-duality should take $J\bar{J}$ to $-J\bar{J}$ and be an isomorphism. From (4.6) one sees that this can only happen if $\bar\epsilon = -\epsilon$ and $\Lambda^{\epsilon,-\epsilon}(R_2) = \Lambda(R_1)$, i.e. if $R_2 = 2/R_1$. Also, up to the action of group-like defects (cf the second line in (4.9)), there is a exactly one T-duality defect operator.

4.1.2. $\hat{R}_2/R_1$ irrational. If $\hat{R}_2/R_1 \notin \mathbb{Q}$ then $\Lambda = \{(0,0)\}$, so that up to a multiplicative constant the defect operators can only consist of the projection $P^{0,0}_{0,0}$ to the vacuum sector. Such an operator corresponds to a defect D with a continuous spectrum of defect fields. Furthermore, $\hat{\bar{D}} \circ \hat{D}$ will decompose into an integral of $\widehat{u}(1)$-preserving fundamental defect operators of $\mathrm{Bos}(R_1)$. Defects of this type probably exist, but it is difficult to check their consistency in detail.

4.2. General topological defects

So far we have considered topological defects that actually preserve the full $\widehat{u}(1)$-symmetry. If we only require that the defect intertwines the Virasoro algebra, i.e. only impose (2.2), but not (4.1), then there are also other defects. In order to understand how they arise, we need to decompose the various $\widehat{u}(1)$-representations $\mathcal{H}_q$ into representations of the Virasoro algebra. The result depends in a crucial manner on the value of q: if q is not an integer multiple of $\frac{1}{\sqrt{2}}$, then $\mathcal{H}_q$ is irreducible with respect to the Virasoro action. On the other hand, if q is an integral multiple of $\frac{1}{\sqrt{2}}$ we have the decomposition

$$\mathcal{H}_{q=\frac{s}{\sqrt{2}}} = \bigoplus_{k=0}^{\infty} \mathcal{H}^{\mathrm{Vir}}_{h=\frac{1}{4}(|s|+2k)^2}, \tag{4.12}$$

where $\mathcal{H}_h^{\text{Vir}}$ denotes the irreducible Virasoro representation of highest weight h (at central charge 1), see e.g. [26]. Whether or not the bulk state space $\mathcal{H}(R)$ contains reducible representations other than $(0, 0)$ depends on the arithmetic properties of R. There are three cases to be distinguished:

Case 1. The equation $x/(2R) + yR = 1/\sqrt{2}$ has no solution for $x, y \in \mathbb{Q}$. (For example, this is the case for $R = 1$.) Then there are no integer solutions to $m/(2R) \pm wR = s/\sqrt{2}$ for any nonzero s. Thus, the only Virasoro-degenerate representations come from the vacuum sector $\{(0, 0)\} = \Lambda(R)$.

Case 2. The equation $x/(2R) + yR = 1/\sqrt{2}$ has a solution with $x, y \in \mathbb{Q}$, and R and $1/R$ are linearly independent over $\mathbb{Q}$. (For example, this is the case for $R = \frac{1}{2}(1 + \sqrt{2})$ which is solved by $x = \frac{1}{4}$, $y = \frac{1}{2}$.) It then follows that the solution x, y is unique. Let L be the least common multiple of the denominators of x and y. Then all integer solutions to $m/(2R) + wR = s/\sqrt{2}$ are of the form $(m, w, s) \in (xL, yL, L)\mathbb{Z}$. Consider the two one-dimensional sub-lattices Λ_l and Λ_r of $\Lambda(R)$ given by

$$\Lambda_l := L\begin{pmatrix} 1/\sqrt{2} \\ x/(2R) - yR \end{pmatrix}\mathbb{Z} \qquad \text{and} \qquad \Lambda_r := L\begin{pmatrix} x/(2R) - yR \\ 1/\sqrt{2} \end{pmatrix}\mathbb{Z}. \tag{4.13}$$

The above calculation shows that all sectors $\mathcal{H}_q \otimes \bar{\mathcal{H}}_{\bar{q}}$ of $\mathcal{H}(R)$ for which $\mathcal{H}_q$ is degenerate are given by $(q, \bar{q}) \in \Lambda_l$, and all sectors for which $\bar{\mathcal{H}}_{\bar{q}}$ is degenerate are given by $(q, \bar{q}) \in \Lambda_r$. Note that $\Lambda_l \cap \Lambda_r = \{(0, 0)\}$.

Case 3. The equation $x/(2R) + yR = 1/\sqrt{2}$ has a solution with $x, y \in \mathbb{Q}$, and R and $1/R$ are linearly dependent over $\mathbb{Q}$, i.e. there are nonzero $u, v \in \mathbb{Q}$ such that $uR + v/R = 0$. Then $R^2 = -v/u$, and hence R^2 is rational. We can then write $R^2 = P/(2Q)$ for some coprime $P, Q \in \mathbb{Z}_{>0}$. Substituting this into $x/(2R) + yR = 1/\sqrt{2}$ we conclude that $x + yP/Q = R/\sqrt{2}$, from which it follows that R is a rational multiple of the self-dual radius $R_{\text{s.d.}} = 1/\sqrt{2}$,

$$R = \frac{E}{F} R_{\text{s.d.}}, \tag{4.14}$$

for some coprime non-negative integers E and F. (In particular, we have $P = E^2$ and $Q = F^2$.) It is now easy to check that (m, w) is an integer solution to $m/(2R) + wR \in 2^{-\frac{1}{2}}\mathbb{Z}$ if and only if $mF^2 + wE^2 \in EF\mathbb{Z}$. Evaluating the latter condition modulo E and modulo F, and using that E and F are coprime, shows that $(m, w) \in (E\mathbb{Z}) \times (F\mathbb{Z})$ provides all solutions to $m/(2R) + wR \in 2^{-\frac{1}{2}}\mathbb{Z}$. The same set also provides all solutions to $m/(2R) - wR \in 2^{-\frac{1}{2}}\mathbb{Z}$. It follows that in a sector $\mathcal{H}_q \otimes \bar{\mathcal{H}}_{\bar{q}}$ of $\mathcal{H}(R)$ either both $\mathcal{H}_q$ and $\bar{\mathcal{H}}_{\bar{q}}$ are degenerate, or neither of them is, and that the Virasoro-degenerate representations occur precisely for $(q, \bar{q}) \in \Lambda$ with

$$\Lambda = \Lambda(R) \cap \Lambda(R_{\text{s.d.}}) = \left\{ \frac{kF}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{lE}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \,\middle|\, k, l \in \mathbb{Z} \right\}. \tag{4.15}$$

In this paper, we shall only analyse case 3. From the results in case 3 one would suspect that any additional defects arising for cases 1 and 2 have a continuous spectrum of defect fields and that their fusion can lead to a continuum of defects, rather than just a discrete superposition, similar to the situation in section 4.1.2. We note in passing that in case 3 the theory is always rational, while case 2 can never arise for rational theories. (The free boson theory is rational iff R^2 is a rational multiple of $R_{\text{s.d.}}^2 = \frac{1}{2}$.) On the other hand, case 1 may or may not be rational.

4.3. Virasoro-preserving defects at the self-dual radius

It is instructive to consider first the simplest example of case 3 which arises when $R = R_{\mathrm{s.d.}}$. The free boson theory is then equivalent to the $su(2)$ WZW model at level 1.[7] The integrable $\widehat{su}(2)_1$-representations can be decomposed into $\widehat{u}(1)$-representations (see, e.g., [27, section 15.6.2]), all of which are in turn reducible with respect to the Virasoro algebra, and thus decompose as in (4.12). The resulting decomposition is most easily understood in terms of the zero modes of the affine Lie algebra which commute with the Virasoro generators. This allows us to decompose the space of states simultaneously with respect to the two Virasoro algebras (generated by L_m and $\bar{L}_m$) and the two commuting $su(2)$ algebras (generated by J_0^a and $\bar{J}_0^a$), leading to (see, for example, [15, section 2])

$$\mathcal{H}(R_{\mathrm{s.d.}}) = \bigoplus_{\substack{s,\bar{s}=0 \\ s+\bar{s}\ \text{even}}}^{\infty} \mathcal{H}_{[s,\bar{s}]}, \qquad \text{where} \quad \mathcal{H}_{[s,\bar{s}]} = V_{s/2} \otimes \bar{V}_{\bar{s}/2} \otimes \mathcal{H}^{\mathrm{Vir}}_{s^2/4} \otimes \bar{\mathcal{H}}^{\mathrm{Vir}}_{\bar{s}^2/4}. \tag{4.16}$$

Here, $\mathcal{H}^{\mathrm{Vir}}_h$ denotes the irreducible Virasoro representation of highest weight h (at central charge 1), while V_j is the irreducible $su(2)$-representation of spin j. The spaces denoted by a barred symbol give the representation of the anti-holomorphic modes.

Because of the condition (2.2), a topological defect has to act as a multiple of the identity on each sector $\mathcal{H}^{\mathrm{Vir}}_{s^2/4} \otimes \bar{\mathcal{H}}^{\mathrm{Vir}}_{\bar{s}^2/4}$. However, since these spaces now appear with multiplicity $\dim(V_{s/2} \otimes \bar{V}_{\bar{s}/2})$, one can have a non-trivial action on these multiplicity spaces. In fact, one finds, in close analogy to the result for conformal boundary conditions [15], that the action of the defect operator on the multiplicity spaces $V_{s/2} \otimes \bar{V}_{\bar{s}/2}$ is described by a pair of group elements $g, h \in SL(2, \mathbb{C})$, where g acts on each $V_{s/2}$ and h on each $\bar{V}_{\bar{s}/2}$. Since the action of $SL(2, \mathbb{C})$ on these spaces is described in terms of J_0^a and $\bar{J}_0^a$, respectively, we can write the defect operator compactly as

$$\hat{D}(g, h) = \exp\left(\alpha_a J_0^a + \beta_b \bar{J}_0^b\right) \qquad \text{where} \quad g = \exp(\alpha_a J^a), \qquad h = \exp(\beta_a J^a). \tag{4.17}$$

More explicitly, on the summand $\mathcal{H}_{[s,\bar{s}]}$ in (4.16) the action is

$$\hat{D}(g, h)\big|_{\mathcal{H}_{[s,\bar{s}]}} = \rho_{s/2}(g) \otimes \rho_{\bar{s}/2}(h) \otimes \mathrm{id}_{\mathcal{H}^{\mathrm{Vir}}_{s^2/4}} \otimes \mathrm{id}_{\bar{\mathcal{H}}^{\mathrm{Vir}}_{\bar{s}^2/4}}, \tag{4.18}$$

where $\rho_{s/2}$ is the representation of $SL(2, \mathbb{C})$ on $V_{s/2}$ and similarly for $\rho_{\bar{s}/2}$. It is then easy to see that $\hat{D}(g, h) = \hat{D}(g', h')$ if and only if $(g, h) = (g', h')$ or $(g, h) = (-g', -h')$ and that the composition of two such defect operators is given by

$$\hat{D}(g, h) \circ \hat{D}(g', h') = \hat{D}(gg', hh'). \tag{4.19}$$

Thus, all fundamental Virasoro-preserving defect operators of $\mathrm{Bos}(R_{\mathrm{s.d.}})$ are group-like and the corresponding group is

$$\mathcal{G}^{\mathrm{Vir}}_{R_{\mathrm{s.d.}}} = \big(SL(2, \mathbb{C}) \times SL(2, \mathbb{C})\big)/\{\pm 1\}. \tag{4.20}$$

Except for the quotient by the centre $\mathbb{Z}_2$ of the diagonally embedded $SL(2, \mathbb{C})$, which is a direct consequence of the restriction to $s + \bar{s} \in 2\mathbb{Z}$ in the sum (4.16), we thus find just a doubling of the result found for conformal boundary states [15], which are parametrized by $SL(2, \mathbb{C})$. In contradistinction to the case of the boundary conditions, the multiplicative structure of $SL(2, \mathbb{C})$ now has a meaning in terms of the fusion of the defects; similarly, the action of a defect operator on a boundary state is given by the adjoint action $D(g_1, g_2)\|h\rangle\rangle = \|g_1 h g_2^{-1}\rangle\rangle$. Using the description in terms of D-branes of the product theory one can prove, using the same methods as in [15], that these are the only fundamental defect operators in this case. We shall give an alternative proof directly in terms of defects in section 6.3.

[7] Note that the $\widehat{su}(2)_1$ modes satisfy $[J^3_m, J^\pm_n] = \pm J^\pm_{m+n}$ and $[J^3_m, J^3_n] = \frac{1}{2} m \delta_{m,-n}$. In particular, the $u(1)$-current $J(z)$ is related to $J^3(z)$ via $J(z) = \sqrt{2} J^3(z)$.

4.4. Virasoro-preserving defects at rational multiples of $R_{\text{s.d.}}$

In the previous section, we have constructed all fundamental defect operators of the self-dual theory. Together with the $\widehat{u}(1)$-preserving defect operators that we found in section 4.1.1, we can now immediately obtain a class of defect operators between two theories whose radii are rational multiples of the self-dual radius $R_{\text{s.d.}}$, i.e.,

$$R_1 = \frac{E_1}{F_1} R_{\text{s.d.}}, \qquad R_2 = \frac{E_2}{F_2} R_{\text{s.d.}}. \tag{4.21}$$

In fact, we can simply compose two $\widehat{u}(1)$-preserving radius-changing defects with a general Virasoro-preserving defect at the self-dual radius, which for the corresponding operators yields

$$\hat{D}(0,0)^{+,+}_{R_2,R_{\text{s.d.}}} \circ \hat{D}(g,h) \circ \hat{D}(0,0)^{+,+}_{R_{\text{s.d.}},R_1} =: \hat{D}(g,h)_{R_2,R_1}. \tag{4.22}$$

We could also use the $\widehat{u}(1)$-preserving defects with $(\epsilon,\bar{\epsilon}) \neq (+,+)$ or with $(x,y) \neq (0,0)$, but as shall become clear below, they do not generate any additional defect operators. In fact, since the Virasoro-degenerate representations in $\mathcal{H}(R_1)$ are precisely those representations that $\mathcal{H}(R_1)$ has in common with $\mathcal{H}(R_{\text{s.d.}})$, it is reasonable to assume that (4.22) does produce all new defect operators that appear when imposing (2.2), but not (4.1). In section 6.3, we present an argument that this is indeed the case.

It turns out that the operators (4.22) are not all distinct. To describe the identification rule, it is helpful to introduce, following [16], the matrices

$$\Gamma_L := \begin{pmatrix} \mathrm{e}^{\pi \mathrm{i}/L} & 0 \\ 0 & \mathrm{e}^{-\pi \mathrm{i}/L} \end{pmatrix} \in SU(2). \tag{4.23}$$

One then finds (see section 6.3 for details) that the operators (4.22) are parametrized by elements of the double coset

$$\mathbb{Z}_{E_2} \times \mathbb{Z}_{F_2} \big\backslash \big(SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\{\pm 1\} \big) \big/ \mathbb{Z}_{E_1} \times \mathbb{Z}_{F_1}, \tag{4.24}$$

where the quotient by $\{\pm 1\}$ is as in (4.20), the right action of an element $(k,l) \in \mathbb{Z}_{E_1} \times \mathbb{Z}_{F_1}$ is as $(g,h) \mapsto \big(g\Gamma^k_{E_1}\Gamma^l_{F_1}, h\Gamma^{-k}_{E_1}\Gamma^l_{F_1}\big)$ and the left action of an element $(k,l) \in \mathbb{Z}_{E_2} \times \mathbb{Z}_{F_2}$ is given by $(g,h) \mapsto \big(\Gamma^k_{E_2}\Gamma^l_{F_2} g, \Gamma^{-k}_{E_2}\Gamma^l_{F_2} h\big)$.

We also observe that the $\widehat{u}(1)$-preserving defect operators at $R_{\text{s.d.}}$ are just special cases of (4.20), for example

$$\hat{D}(x,y)^{+,+}_{R_{\text{s.d.}},R_{\text{s.d.}}} = \hat{D}(g,h)$$
$$\text{for} \quad g = \begin{pmatrix} \mathrm{e}^{\pi \mathrm{i}\sqrt{2}x} & 0 \\ 0 & \mathrm{e}^{-\pi \mathrm{i}\sqrt{2}x} \end{pmatrix}, \qquad h = \begin{pmatrix} \mathrm{e}^{-\pi \mathrm{i}\sqrt{2}y} & 0 \\ 0 & \mathrm{e}^{\pi \mathrm{i}\sqrt{2}y} \end{pmatrix}, \tag{4.25}$$

see section 6.3 for more details. In particular, $D(\Gamma_L, \Gamma_{L'}) = D\big(\frac{1}{\sqrt{2}L}, -\frac{1}{\sqrt{2}L'}\big)^{+,+}_{R_{\text{s.d.}},R_{\text{s.d.}}}$.

The composition law for these defect operators can then be easily obtained by combining (4.9), (4.25) and (4.19). One finds

$$\hat{D}(g,h)_{R_3,R_2} \circ \hat{D}(g',h')_{R_2,R_1} = \sum_{m=0}^{E_2-1} \sum_{w=0}^{F_2-1} \hat{D}\big(g\Gamma^m_{E_2}\Gamma^w_{F_2}g', h\Gamma^{-m}_{E_2}\Gamma^w_{F_2}h'\big)_{R_3,R_1}. \tag{4.26}$$

Using this result we can verify for which (g,h) the defect operator $\hat{D}(g,h)_{R_2,R_1}$ is fundamental. To this end, we need to check whether the trivial defect appears exactly once in the product $\hat{D}\big(g^{-1},h^{-1}\big)_{R_1,R_2} \circ \hat{D}(g,h)_{R_2,R_1}$, as is required for fundamental defect operators. To do so, we count for how many pairs (m,w) the elements $g^{-1}\Gamma^m_{E_2}\Gamma^w_{F_2}g$ and $h^{-1}\Gamma^{-m}_{E_2}\Gamma^w_{F_2}h$ are equal to (e,e) modulo the identification (4.24). This in particular requires $g^{-1}\Gamma^m_{E_2}\Gamma^w_{F_2}g$ (say) to be diagonal, which in turn is possible only if $(m, w) = (0, 0)$ or if g is itself either diagonal or anti-diagonal. The same holds for h. Thus if g and h are neither diagonal nor anti-diagonal, then $\hat{D}(g, h)_{R_2,R_1}$ is fundamental. On the other hand, if g and h are diagonal or anti-diagonal, then whether $\hat{D}(g, h)_{R_2,R_1}$ is fundamental or not depends on the divisibility properties of R_1 and R_2. For example, if g and h are both diagonal and if E_1, E_2 are coprime and F_1, F_2 are coprime, then $\hat{D}(g, h)_{R_2,R_1}$ is fundamental. On the other hand, if g and h are as in (4.25) and $R_1 = R_2$, (4.9) implies that

$$\hat{D}(g, h)_{R_1,R_1} = \sum_{m=0}^{F_1-1} \sum_{w=0}^{E_1-1} \hat{D}\left(\frac{m}{\sqrt{2}E_1} + \frac{w}{\sqrt{2}F_1} + x, \frac{m}{\sqrt{2}E_1} - \frac{w}{\sqrt{2}F_1} + y\right)^{+,+}_{R_1,R_1}. \tag{4.27}$$

Thus it follows that for $R_1 \neq R_{\text{s.d.}}$, $\hat{D}(g, h)_{R_1,R_1}$ is not fundamental if g and h are both diagonal.

4.5. Completeness of the set of defect operators

Before proceeding to the detailed calculations, let us discuss whether the results listed in sections 4.1–4.4 provide all topological defects for the free boson. There are in fact three related, but distinct, questions one can pose. The first and most obvious one is

Q1. What are all topological defects that join the theories $\text{Bos}(R_1)$ and $\text{Bos}(R_2)$?

Just as in the rational case treated in section 5, the collection of all topological defects is probably best described as a suitable category, rather than as a set. To address Q1 one must then decide when two defects should be regarded as 'isomorphic', i.e. when they give rise to the same correlators involving in particular defect fields, but also bulk fields, boundaries, etc. While this will in general be difficult, one can restrict oneself to considering defect operators and ask the more concrete question

Q2. What are all operators $L : \mathcal{H}(R_1) \to \mathcal{H}(R_2)$ that arise as defect operator for some topological defect joining $\text{Bos}(R_1)$ and $\text{Bos}(R_2)$?

The questions Q1 and Q2 are indeed different: in general a defect is not determined uniquely by its associated defect operator. For example, while the operator does determine the spectrum of defect fields, it is not always possible to deduce their OPE or even the representation of the Virasoro algebra on the space of defect fields. This aspect can be stressed by asking instead

Q2'. Given an operator $L : \mathcal{H}(R_1) \to \mathcal{H}(R_2)$, what are all topological defects D joining $\text{Bos}(R_1)$ and $\text{Bos}(R_2)$ such that $L = \hat{D}$?

As an illustration, we construct in appendix A a one-parameter family of mutually distinct defects which all have the same defect operator. This family is obtained by perturbing a superposition of two defects by a marginal defect-changing field. In this example, the perturbing field is not self-adjoint, and the resulting defects are 'logarithmic' in the sense that the perturbed Hamiltonian is no longer diagonalizable. One can then ask whether restricting oneself to defects for which the Hamiltonian generating translations along the cylinder is self-adjoint on all state spaces (for disorder-, defect- or defect-changing fields) makes the assignment (defect) $\mapsto$ (defect operator) injective. We think that this is indeed true for the free boson, but we do not have a proof.

For the Virasoro-preserving defects, in this paper we will consider the following variant of Q2:

Q3. What are all operators $L : \mathcal{H}(R_1) \to \mathcal{H}(R_2)$ that arise as defect operator for some topological defect D, for which the spectrum of defect fields (calculated from the torus with insertion of $\bar{D} * D$) contains a unique (up to normalization) state of lowest conformal weight $h = \bar{h} = 0$, separated by a gap from the rest of the spectrum with $h + \bar{h} > 0$?

Uniqueness of the lowest weight state implies that the defect operator $\hat{D}$ is fundamental. For, suppose that $\hat{D} = \hat{D}_1 + \hat{D}_2$. Then the trace contains terms coming from $\bar{D}_1 * D_1$ and $\bar{D}_2 * D_2$, both of which lead to a field with conformal weight $h = \bar{h} = 0$ in the spectrum.

In section 6.3 (as summarized in section 4.3), we answer Q3 for $\mathrm{Bos}(R_{\mathrm{s.d.}})$ by showing that there cannot be more defect operators than those listed in (4.18) and that they in fact are all realized as perturbations of the trivial defect. Furthermore, we argue that the answer to Q3 for $\mathrm{Bos}(R)$, with R a rational multiple of $R_{\mathrm{s.d.}}$, is given by combining the defect operators (4.6) (with $R_1 = R_2 = R$) and (4.22). We lack a proof that these are all possible operators (see, however, the short remark at the end of section 6.3).

Finally, we would like to stress that the same issues we discussed above for topological defects also arise in the classification of conformal boundary conditions. Indeed, it is in general not true that the boundary state determines the boundary condition uniquely. Again an example can be constructed by perturbing a superposition of boundary conditions by a (non-selfadjoint) boundary changing field.

5. Free boson with extended chiral symmetry

In this section, we investigate topological defects for the compactified free boson using the methods developed in [7, 11, 28]. These apply to *rational* conformal field theories.

5.1. Chiral symmetry

For any $N \in \mathbb{Z}_{>0}$, the $u(1)$ current algebra can be extended by the two vertex operators $W_N^{\pm}(z) = :\mathrm{e}^{\pm 2\mathrm{i}\sqrt{2N}\phi(z)}:$ of $u(1)$-charge $\pm\sqrt{2N}$ and conformal weight N. The resulting chiral algebra, denoted by $\widehat{u}(1)_N$, is in fact rational. $\widehat{u}(1)_N$ has $2N$ inequivalent irreducible highest weight representations, labelled $U_0, U_1, \ldots, U_{2N-1}$, which decompose into irreducible representations of the $u(1)$ (vertex) subalgebra as

$$U_k \cong \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_{(k+2Nm)/\sqrt{2N}}. \tag{5.1}$$

We denote by $\mathcal{U}_N$ the category formed by the representations of $\widehat{u}(1)_N$. $\mathcal{U}_N$ is a modular tensor category; thus it is balanced braided rigid monoidal, which means, roughly speaking, that given two objects U, V of $\mathcal{U}_N$ one can take the tensor product $U \otimes V$ (monoidal), that there are two distinct ways to go from $U \otimes V$ to $V \otimes U$ (braided), that each object has a two-sided dual, namely its contragredient representation (rigid) and a balancing twist, namely its exponentiated conformal weight (balanced); and finally the braiding is maximally non-degenerate (modular). See, e.g., [29, appendix A.1] for details and references.

5.2. Rational compactified free boson

In the TFT approach to rational CFT [7] one finds that the algebra of boundary fields for a given single boundary condition determines the entire CFT, including in particular the bulk spectrum, other boundary conditions and also topological defects. The boundary conditions and topological defects one finds in this way are precisely those which preserve the rational chiral algebra.

In terms of the representation category, the algebra of boundary fields of the rational free boson theory is a symmetric special Frobenius algebra A in the category $\mathcal{U}_N$. If one requires that there is a unique boundary vacuum state, i.e. a unique primary boundary field of conformal weight zero, then, as shown in [28, section 3.3], every such algebra is of the form

$$A_r = \bigoplus_{k=0}^{r-1} U_{2kN/r} \qquad \text{where} \quad r \in \mathbb{Z}_{>0}, \qquad r \text{ divides } N, \tag{5.2}$$

and for each of them the multiplication is unique up to isomorphism. In other words, together with (5.1), A_r gives the space of boundary fields, and the boundary OPE is unique up to field redefinition.

All other quantities of the full CFT can be computed starting from the algebra A_r. Consider for example the matrix $Z_{ij}^{(r)}$ which determines the bulk partition function via $Z = \sum_{i,j=1}^{2N-1} Z_{ij}^{(r)} \chi_i(\tau)\chi_j(-\tau^*)$. Here $\chi_k(\tau) = \mathrm{tr}_{U_k} \mathrm{e}^{2\pi\mathrm{i}\tau(L_0-\frac{1}{24})}$ is the Virasoro character of U_k. For A_r one finds [7, section 5.6.1][8]

$$Z_{ij}^{(r)} = \delta_{i+j,0}^{[2N/r]} \delta_{i,j}^{[2r]}, \tag{5.3}$$

where $\delta^{[p]}$ is the periodic Kronecker symbol, i.e. $\delta_{i,j}^{[p]} = 1$ if $i \equiv j \bmod p$ and $\delta_{i,j}^{[p]} = 0$ else. Comparison with (3.2) and (5.1) shows that the compactification radius R is related to r and N via

$$R = r/\sqrt{2N}. \tag{5.4}$$

Conversely, the free boson compactified at a radius of the form $R = \sqrt{P/(2Q)}$ with P, Q coprime positive integers contains in its subset of holomorphic bulk fields the chiral algebra $\widehat{u}(1)_{n^2PQ}$, for any choice of $n \in \mathbb{Z}_{>0}$. The relevant algebra in $\mathcal{U}_{n^2PQ}$ is then A_r with $r = nP$.

The simplest choice is just to set $n = 1$. However, recall that in the TFT approach one can only describe boundary conditions and topological defects that preserve the rational chiral algebra one selects as a starting point. If we keep n as a parameter, this chiral algebra is $\widehat{u}(1)_{n^2PQ}$. Taking n large allows us to obtain more boundary conditions and defects. In fact, the $\widehat{u}(1)$ algebra and its extension $\widehat{u}(1)_{n^2PQ}$ start to differ only from L_0-eigenvalue n^2PQ onwards, and the explicit calculation shows that in the $n \to \infty$ limit we obtain all boundary conditions and defects that preserve only the $u(1)$ current algebra.

5.3. *Rational topological defects as bimodules*

Let us fix a chiral algebra $\widehat{u}(1)_N$. We want to compute the topological defects that interpolate between the free boson CFTs described by two algebras A_r and A_s in $\mathcal{U}_N$, i.e. those topological defects that are transparent to the fields of $\widehat{u}(1)_N$ and link a free boson compactified at radius $R_1 = r/\sqrt{2N}$ to a free boson compactified at $R_2 = s/\sqrt{2N}$. Note that in this description we automatically have $R_1/R_2 \in \mathbb{Q}$.

In the TFT approach the task of finding all topological defects preserving $\widehat{u}(1)_N$ is reduced to finding all A_r–A_s-bimodules in the category $\mathcal{U}_N$. We will solve the latter problem in two steps. First we describe all A_r-modules and A_r-bimodules, and afterwards also the A_r–A_s-bimodules for $r \neq s$.

[8] The notation A_{2r} used in [7] corresponds to $A_{N/r}$ in the present convention, e.g. here $A_1 = U_0$, while in [7] one has $A_{2N} = U_0$.

5.3.1. Modules and bimodules of the algebra A_r. Since the objects U_j appearing in the decomposition (5.2) of A_r are all simple currents [30], the modules and bimodules of A_r can be obtained using the methods of [11, 28].

As for the left A_r-modules, one finds that every simple module is isomorphic to an induced module $A_r \otimes U_k$ and that the induced modules $A_r \otimes U_k$ and $A_r \otimes U_l$ are isomorphic if and only if $k \equiv l \bmod 2N/r$. Thus there are $2N/r$ distinct simple (left) A_r-modules, which we denote by $M_\kappa^{(r)}, \kappa \in \{0, 1, \ldots, \frac{2N}{r} - 1\}$. The one-point function of a bulk field on a disc with boundary condition labelled by $M_\kappa^{(r)}$ is zero for a pure momentum state, so that in string theory these boundary conditions correspond to $D1$-branes with equally spaced Wilson line parameter.

At this point it is useful to think of the radius as being given and of the form $R = \sqrt{P/(2Q)}$ (with P and Q coprime), and to describe this CFT via the algebra A_{nP} in $\mathcal{U}_{n^2PQ}$, i.e. we set $r = nP$ and $N = n^2PQ$. Then, the label κ of the simple modules takes values in $\{0, 1, \ldots, 2nQ - 1\}$, and the corresponding $D1$-branes have values for their Wilson lines that are equally spaced.

The simple A_r-bimodules can be obtained by the methods of [11, section 5]; we defer the details to appendix B.3. The result is that the (isomorphism classes of) simple A_r-bimodules are in one-to-one correspondence to elements of the Abelian group

$$G_{n,P,Q} = (\mathbb{Z}_{nP} \times \mathbb{Z}_{nQ} \times \mathbb{Z})/\langle(1, 1, -2)\rangle, \tag{5.5}$$

where $\langle(1, 1, -2)\rangle$ denotes the subgroup generated by the element $(1, 1, -2)$ of $\mathbb{Z}_{nP} \times \mathbb{Z}_{nQ} \times \mathbb{Z}$. Note that since every element of $G_{n,P,Q}$ can be written either as $(a, b, 0)$ or $(a, b, 1)$ for suitable a and b, $G_{n,P,Q}$ has $2n^2PQ = 2N$ elements (and thus is in particular finite). We denote these simple A_r-bimodules by $B^{(r)}_{(a,b,\rho)}$ for $(a, b, \rho) \in G_{n,P,Q}$.

The fusion of two topological defects is obtained by computing the tensor product over A_r for the corresponding bimodules. The calculation is done in appendix B.3. The result is that the tensor product is just addition in $G_{n,P,Q}$, i.e. for any two elements $(a, b, \rho), (c, d, \sigma) \in G_{n,P,Q}$ one has

$$B^{(r)}_{(a,b,\rho)} \otimes_{A_r} B^{(r)}_{(c,d,\sigma)} \cong B^{(r)}_{(a+c,b+d,\rho+\sigma)}. \tag{5.6}$$

In particular, all topological defects preserving the extended chiral symmetry $\widehat{u}(1)_N$ are group-like. Similarly, the fusion of a topological defect to a conformal boundary condition is obtained by the tensor product over A_r. One finds

$$B^{(r)}_{(a,b,\rho)} \otimes_{A_r} M^{(r)}_\kappa \cong M^{(r)}_{\kappa+2b+\rho}. \tag{5.7}$$

Thus, the topological defect labelled by $B_{(a,b,\rho)}$ shifts the Wilson line of the $D1$-branes by $2b + \rho$ units. As shown by (5.8), the parameter a amounts to a similar shift in the position of the $D0$-branes. Of course, $D0$-branes, for which in the present convention $J(x) = -\bar{J}(x)$ on the boundary $x \in \mathbb{R}$ of the upper half plane, do not appear in the present description, as they do not preserve $\widehat{u}(1)_N$. But we can still deduce the effect of a by computing the action of topological defects on bulk fields.

Let $\phi_{q,\bar{q}}$ be a bulk field corresponding to a state in the sector $\mathcal{H}_q \otimes \bar{\mathcal{H}}_{\bar{q}}$ of the space (3.2) of bulk states. The action of the topological defect $D^{(r)}_{(a,b,\rho)}$ corresponding to the bimodule $B^{(r)}_{(a,b,\rho)}$ amounts simply to the multiplication by a phase (see appendix B.3):

$$\hat{D}^{(r)}_{(a,b,\rho)}\phi_{q,\bar{q}} = \exp\left\{-2\pi \mathrm{i}\left(\frac{a+\rho/2}{nP}R(q+\bar{q}) + \frac{b+\rho/2}{nQ}\frac{1}{2R}(q-\bar{q})\right)\right\}\phi_{q,\bar{q}}. \tag{5.8}$$

We see that the parameter a gives a phase shift depending only on the total momentum $q + \bar{q}$ of $\phi_{q,\bar{q}}$, while b gives a phase shift depending on the winding number $q - \bar{q}$. Also, in the limit of large n we find that the phase shifts are parametrized by two continuous parameters taking values in $\mathbb{R}/\mathbb{Z}$. This agrees with the result stated in section (4.6); specifically,

$$\hat{D}^{(r)}_{(a,b,\rho)} = \hat{D}(-(a+b+\rho)/\sqrt{2N}, (a-b)/\sqrt{2N})^{+,+}_{R,R}, \tag{5.9}$$

where $R = r/\sqrt{2N}$. This also shows that at least for these values of the parameters, the operators $\hat{D}(x,y)^{+,+}_{R,R}$ are indeed defect operators for a consistent defect.

5.3.2. A_r–A_s-bimodules and radius-changing defects. Consider two algebras A_r and A_s in $\mathcal{U}_N$. As shown in appendix B.4, all simple A_r–A_s-bimodules can be obtained as follows. The algebra $A_{\mathrm{lcm}(r,s)}$ is the smallest algebra that has both A_r and A_s as a subalgebra. By embedding A_r and A_s into $A_{\mathrm{lcm}(r,s)}$ one defines the structure of an A_r–A_s-bimodule on $A_{\mathrm{lcm}(r,s)}$; we denote this bimodule by $A^{(rs)}$. Every simple A_r–A_s-bimodule $B^{(rs)}$ can then be written as

$$B^{(rs)} \cong B^{(r)}_{(a,b,\rho)} \otimes_{A_r} A^{(rs)} \cong A^{(rs)} \otimes_{A_s} B^{(s)}_{(c,d,\sigma)}, \tag{5.10}$$

for $B^{(r)}_{(a,b,\rho)}$ and $B^{(s)}_{(c,d,\sigma)}$ appropriate A_r- and A_s-bimodules, respectively. The total number of inequivalent simple A_r–A_s-bimodules is given by $\mathrm{tr}(Z^{(r)} Z^{(s)}) = 2\mathrm{gcd}(r,s)\mathrm{gcd}\big(\frac{N}{r}, \frac{N}{s}\big)$, where $Z^{(\cdot)}$ is the matrix (5.3), see [7, remark 5.19].

Let us refer to a defect as being elementary iff it corresponds to a simple bimodule. The result above can then be rephrased as the statement that there are $2\mathrm{gcd}(r,s)\mathrm{gcd}\big(\frac{N}{r}, \frac{N}{s}\big)$ distinct elementary topological defects (transparent to fields in the chiral algebra $\widehat{u}(1)_N$) that join the free boson compactified at radii $r/\sqrt{2N}$ and $s/\sqrt{2N}$. All of these can be written as the fusion $D^{(r)}_{(a,b,\rho)} * D^{(rs)}$ or $D^{(rs)} * D^{(s)}_{(c,d,\sigma)}$, where $D^{(rs)}$ is the defect corresponding to the bimodule $A^{(rs)}$. In particular, all such radius-changing topological defects lie on a single orbit with respect to the action of the group-like defects of the free boson on either side of the radius-changing defect.

As we did for the defects $D^{(r)}_{(a,b,\rho)}$, let us also give the defect operator for the topological defect $D^{(rs)}$. One finds that, up to an overall constant, $\hat{D}^{(rs)}$ projects onto fields with left/right $u(1)$-charges in the intersection of the charge lattices of the two theories. Concretely, if we fix a primary bulk field $\phi^{(r)}_{x,y}$ in each sector[9] $U_x \otimes \overline{U}_y$ of $\mathcal{H}\big(R = \frac{r}{\sqrt{2N}}\big)$ as in appendix B.4, we have

$$\hat{D}^{(rs)} \phi^{(s)}_{x,y} = \frac{\mathrm{lcm}(r,s)}{r} \delta^{[2N/\mathrm{gcd}(r,s)]}_{x+y,0} \delta^{[2\mathrm{lcm}(r,s)]}_{x,y} \phi^{(r)}_{x,y}. \tag{5.11}$$

The relation to the operator given in (4.6) is[10] $\hat{D}^{(rs)} = \sqrt{s/r}\,\hat{D}(0,0)^{+,+}_{R_2,R_1}$, where $R_1 = s/\sqrt{2N}$ and $R_2 = r/\sqrt{2N}$.

As opposed to the general non-rational case (compare the discussion in section 4.5), for defects that preserve the rational chiral algebra one can show that the defect operator determines the defect uniquely, in the sense that it fixes the corresponding bimodule up to isomorphism [11, proposition 2.8]. For example, using (5.11) together with $D^{(rs)} * D^{(sr)} = D^{(r)}_{A^{(rs)} \otimes_{A_s} A^{(sr)}}$ it is straightforward to check that

$$A^{(rs)} \otimes_{A_s} A^{(sr)} \cong \bigoplus_{m=0}^{\frac{\tilde{\ell}}{N/r}-1} \bigoplus_{n=0}^{\frac{\ell}{r}-1} D^{(r)}_{(\frac{N}{\tilde{\ell}}m, \frac{N}{\ell}n, 0)} \qquad \text{with} \qquad \tilde{\ell} = \mathrm{lcm}\left(\frac{N}{r}, \frac{N}{s}\right), \quad \ell = \mathrm{l\,cm}\,(r,s). \tag{5.12}$$

[9] Note that here the symbol '$\otimes$' does not stand for the tensor product of the category $\mathcal{C}$, but rather $U \otimes \overline{V}$ is an object of the product category (see [31]) $\mathcal{C} \boxtimes \overline{\mathcal{C}}$.

[10] The additional factor arises from different normalization conventions for the zero-point functions on the sphere. The expression (4.6) is computed for $\langle \mathbf{1}^{(R_2)} \rangle / \langle \mathbf{1}^{(R_1)} \rangle = 1$, whereas the TFT approach selects $\langle \mathbf{1}^{(R_2)} \rangle / \langle \mathbf{1}^{(R_1)} \rangle = s/r$. The analogue of (4.6) with unfixed normalizations is given in (6.20).

Acting with the defect $D^{(rs)}$ on a boundary condition of the free boson at radius $s/\sqrt{2N}$ results in a boundary condition of the free boson at radius $r/\sqrt{2N}$. It is easy to compute the corresponding tensor product $A^{(rs)} \otimes_{A_s} M_\kappa^{(s)} \cong A^{(rs)} \otimes_{A_s} A_s \otimes U_\kappa \cong A^{(rs)} \otimes U_\kappa$. Comparing the respective decompositions into simple objects of $\mathcal{U}_N$ one finds that

$$A^{(rs)} \otimes_{A_s} M_\kappa^{(s)} \cong \bigoplus_{m=0}^{\frac{N/r}{\tilde{g}}-1} M_{\kappa+2m\tilde{g}}^{(r)} \qquad \text{with} \quad \tilde{g} = \gcd\left(\frac{N}{r}, \frac{N}{s}\right) \tag{5.13}$$

as A_r-modules.

From (5.11) it is easy to see that $D^{(rs)}$ cannot give an equivalence of theories unless $r = s$, in which case $\hat{D}^{(rs)}$ is the identity. For $r \neq s$, $\hat{D}^{(rs)}$ will map some of the bulk fields to zero.

5.4. T-duality

On the CFT level, T-duality amounts to the statement that the free boson CFTs at radius R and $R' = \alpha'/R \equiv 1/(2R)$ are isomorphic; the isomorphism inverts the sign of $J\bar{J}$. To find a defect which implements this isomorphism, it is thus not sufficient to look only among the defects transparent to the $u(1)$ currents, as was done in section 5.3. Instead, we work with the chiral algebra $\widehat{u}(1)/\mathbb{Z}_2$, which consists of fields invariant under $J \mapsto -J$, and its rational extension $\widehat{u}(1)_N/\mathbb{Z}_2$ by a field of conformal weight N. Denote the category of representations of $\widehat{u}(1)_N/\mathbb{Z}_2$ by $\mathcal{D}_N$.

Let us treat the case $R = 1/\sqrt{2N}$ and $R' = \sqrt{N/2}$ as an example. The details for how to arrive at the statements below can be found in appendix C. $\mathcal{D}_N$ contains two algebras $\tilde{A}_1$ and $\tilde{A}_N$ which describe the compactifications to R and R', respectively. The algebras $\tilde{A}_1$ and $\tilde{A}_N$ are in fact Morita equivalent, i.e. there exist an $\tilde{A}_1$–$\tilde{A}_N$-bimodule X and an $\tilde{A}_N$–$\tilde{A}_1$-bimodule X', both coming from the twisted sector of the orbifold (see (C.8)), such that one has isomorphisms

$$X \otimes_{\tilde{A}_N} X' \cong \tilde{A}_1 \qquad \text{and} \qquad X' \otimes_{\tilde{A}_1} X \cong \tilde{A}_N \tag{5.14}$$

of bimodules. The corresponding defects D_X and $D_{X'}$ interpolating between these two T-dual CFTs obey

$$\hat{D}_X \hat{D}'_X \phi = \hat{D}_{X \otimes_{\tilde{A}_N} X'} \phi = \hat{D}_{\tilde{A}_1} \phi = \phi \tag{5.15}$$

for any bulk field ϕ of the $\tilde{A}_1$-theory, and vice versa, so that $\hat{D}_X$ indeed provides an isomorphism between the bulk state spaces of the two CFTs.

To relate the correlators of the two compactifications we can use the result [11, section 3.3] that

$$\mathrm{Cor}_{\tilde{A}_1}(\Sigma) = \gamma^{-\chi(\Sigma)} \mathrm{Cor}_{\tilde{A}_N}(\Sigma') \qquad \text{with} \quad \gamma = \dim(\tilde{A}_N)/\dim(X) = \sqrt{N}. \tag{5.16}$$

Here Σ denotes a world sheet decorated with data for the CFT described by $\tilde{A}_1$, i.e. the free boson theory at radius $R = 1/\sqrt{2N}$. This world sheet may have a boundary, as well as insertions of bulk and boundary fields. The world sheet Σ' is equal to Σ as a surface, but it is decorated with data for the free boson compactified at radius R', and the field insertions, boundary conditions and defect lines of Σ are replaced by those obtained via the action of the interpolating defect X'. By $\mathrm{Cor}_{\tilde{A}_1}$ and $\mathrm{Cor}_{\tilde{A}_N}$ we then mean the correlators for the respective world sheets. In the prefactor $\gamma^{-\chi(\Sigma)}$, $\chi(\Sigma)$ is the Euler character of Σ, while γ is the quotient of quantum dimensions for $\mathcal{D}_N$ as stated in the formula.

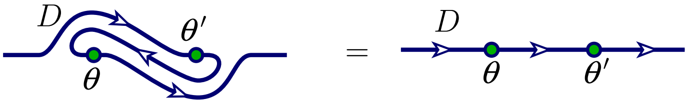
Since in the perturbative expansion of the string free energy, the CFT correlator $\mathrm{Cor}_{\tilde{A}_1}(\Sigma)$ appears with the prefactor $g_s^{-\chi(\Sigma)}$, we see that on the right-hand side of the equality (5.16) the combination $(g_s\gamma)^{-\chi(\Sigma)}$ appears. In the present example we have $\gamma = \sqrt{N} = 1/(\sqrt{2}R)$, so that we obtain precisely the expected identity (3.6). Note that the derivation of the change in g_s in terms of defects can be carried out entirely on the level of the world sheet CFT, without explicitly mentioning the dilaton field.

5.5. *Truly marginal deformations*

One limitation when working within rational CFT is the absence of continuous moduli. That is, there are no continuous families of CFTs with a fixed rational chiral algebra, and no continuous families of boundary conditions or topological defects preserving this chiral algebra [28, section 3.1]. However, one can deduce the existence of moduli by looking for truly marginal fields.

For a boundary field ψ there is a simple sufficient criterion to ensure that it leads to a truly marginal perturbation: ψ needs to have conformal weight one and be *self-local* [32], i.e., exchanging the position of two adjacent boundary fields $\psi(x)\psi(y)$ by analytic continuation in x and y does not modify the value of a correlator. Via the folding trick [13], which relates conformal defects of one CFT to conformal boundary conditions of the product theory, this results in a corresponding condition for defect fields. One obtains in this way a sufficient criterion for a conformal defect to stay conformal under a perturbation. We are more specifically interested in a condition for a topological defect to stay topological. This leads to the following definition.

Let D be a topological defect and let $\mathcal{H}_D^{(1,0)}$ and $\mathcal{H}_D^{(0,1)}$ be the spaces of all defect fields living on the defect D of left/right conformal weight $(1,0)$ and $(0,1)$, respectively. A subspace $\mathcal{L}$ of $\mathcal{H}_D^{(1,0)} \oplus \mathcal{H}_D^{(0,1)}$ is called *self-local* iff for all defect fields $\theta, \theta' \in \mathcal{L}$ we have



(5.17)

inside every correlator. In other words, exchanging the order of θ and θ' along the defect D is equivalent to analytic continuation (in any correlator) of θ past θ'. Note that the vertically reflected version of (5.17) holds as well, as can be seen by simultaneously moving the insertion points of θ and θ' on both sides of the equality such that the defect on the left-hand side becomes a straight line.

Using the same regularization procedure as in [32] one can check that a perturbation of D by a defect field $\theta \in \mathcal{L}$ is again a topological defect.

Let us now consider the free boson compactified at radius $R = \sqrt{P/(2Q)}$ in terms of the algebra A_{nP} in $\mathcal{U}_N$, $N = n^2PQ$. We will also assume $n > 1$ (this avoids the special cases $N = 1, 2$ which require a separate treatment). The three representations of $\widehat{u}(1)_{n^2PQ}$ that contain states of weight one are U_0, U_{2a} and $U_{2(N-a)}$ with $a = n\sqrt{PQ}$. Since P and Q are coprime, a can only be an integer if $P = E^2$ and $Q = F^2$ for some coprime $E, F \in \mathbb{Z}_{>0}$.

Suppose the space of defect fields living on a topological defect D contains a field θ in the sector $U_{2a} \otimes \overline{U}_0$ and suppose that θ has weight $(1,0)$. Then the subspace $\mathbb{C}\theta$ is *always* self-local (see appendix B.5). The same holds for fields of overall weight one in $U_0 \otimes \overline{U}_0$, $U_{2(N-a)} \otimes \overline{U}_0$, $U_0 \otimes \overline{U}_{2a}$ and $U_0 \otimes \overline{U}_{2(N-a)}$. In other words, every field in $\mathcal{H}_D^{(1,0)} \oplus \mathcal{H}_D^{(0,1)}$ with well-defined $u(1)$-charge gives rise to a truly marginal perturbation that leaves D topological.

However, if we perturb a defect D by a field of definite nonzero $u(1)$-charge, the resulting theory is no longer unitary. To see this consider a cylinder $S^1 \times \mathbb{R}$ with the defect D inserted on the line $\{\alpha\} \times \mathbb{R}$ for some $\alpha \in S^1$. The perturbation by a defect field θ amounts to an insertion of $\exp\left(\lambda \int_{-\infty}^{\infty} \theta(\alpha, x)\, \mathrm{d}x\right)$ for some $\lambda \in \mathbb{R}$. The Hamiltonian generating translations along the cylinder is $H(\lambda) = H_0 + \lambda\theta(\alpha, 0)$, where H_0 is the unperturbed Hamiltonian. Since we start from a unitary theory, we have $H_0^\dagger = H_0$. For the perturbed Hamiltonian $H(\lambda)$ to be self-adjoint we need the perturbing operator $\theta(\alpha, 0)$ to be self-adjoint. However, if θ has left/right $u(1)$-charge $(\sqrt{2}, 0)$, say, then $\theta^\dagger$ has charge $(-\sqrt{2}, 0)$. Thus we need to perturb by appropriate linear combinations of defect fields of charges $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

Incidentally, perturbing a topological defect (or a conformal boundary condition) by a marginal field of left and right $u(1)$-charges $(\sqrt{2}, 0)$, say, leads to a *logarithmic* theory, i.e. the perturbed Hamiltonian is no longer diagonalizable. For example, in [33], the operator $I_\lambda(a, b) = \exp\big(\frac{\lambda}{2\pi}\int_a^b J^+(x)\,\mathrm{d}x\big)$ was considered for the $\widehat{su}(2)_1$ WZW model. This operator can be understood as a topological defect running from a to b, obtained by a perturbation of the trivial defect by the field $J^+(z)$. Correlators of $I_\lambda(a, b)$ are found to contain logarithms.

The same will happen, e.g., for conformal boundary conditions of the free boson at the self-dual radius. As shown in [15], these are parametrized by elements of $SL(2, \mathbb{C})$. Correlators involving boundary changing fields that join boundary conditions belonging to different $SU(2)$ cosets in $SL(2, \mathbb{C})$ may contain logarithms. Another example of a non-logarithmic bulk theory which allows for boundary fields with logarithmic correlators was given in [34].

To obtain unitary defect perturbations we thus need to find a self-local subspace that is pointwise fixed with respect to charge conjugation. Rather than trying to classify all such cases, we will consider as a particular example the decomposition of $\hat{D}(g, h)$ for $g = h = e$ as obtained from (4.27). Let D be the defect given by the superposition

$$D = \sum_{k=0}^{E-1}\sum_{l=0}^{F-1} D^{(nP)}_{(ka,la,0)}. \tag{5.18}$$

Note that $ka \equiv 0 \bmod nP$ is equivalent to $knEF \equiv 0 \bmod nE^2$, i.e. $k \equiv 0 \bmod E$, as E and F are coprime. For the same reason, $la \equiv 0 \bmod nQ$ is equivalent to $l \equiv 0 \bmod F$; hence all the elementary defects appearing in the sum (5.18) are distinct.

The space of defect fields living on D is given by (see appendix B.5)

$$\mathcal{H}_D = \bigoplus_{i,j=0}^{2N-1} (U_i \otimes \overline{U}_j)^{\oplus Z^D_{ij}} \qquad \text{with} \quad Z^D_{ij} = EF\delta^{[2nF]}_{i+j,0}\delta^{[2nE]}_{i-j,0}. \tag{5.19}$$

Since $\mathcal{H}_D$ contains the representations $U_0 \otimes U_0$, $U_{2a} \otimes U_0$, etc, with multiplicity EF, the space $\mathcal{H}_D^{(1,0)} \oplus \mathcal{H}_D^{(0,1)}$ has dimension $6EF$. It contains a six-dimensional self-local subspace $\mathcal{L}$ pointwise fixed under charge conjugation. The subspace $\mathcal{L}$ is not unique, but maximal in the sense that there is no self-local subspace of $\mathcal{H}_D^{(1,0)} \oplus \mathcal{H}_D^{(0,1)}$ of which $\mathcal{L}$ is a proper subspace (see appendix B.5).

This is in accordance with the results summarized in section 4.4, where for a rational multiple of the self-dual radius a six-dimensional moduli space of topological defects is found. The defect operators described there are fundamental, except possibly when g and h are diagonal or anti-diagonal. In particular, for $(g, h) = (e, e)$ on obtains the superposition (5.18).

6. General topological defects for the free boson

In this section, we present details of the calculations that lead to the results stated in section 4 for topological defects joining two free boson CFTs $\mathrm{Bos}(R_1)$ and $\mathrm{Bos}(R_2)$. We first give the relation between the defect operators $\hat{D}$ and $\hat{\hat{D}}$ (section 6.1). Then we consider defects that preserve the $\widehat{u}(1)$-symmetry up to an automorphism (section 6.2). In section 6.3, we investigate defects that preserve only the Virasoro algebra for the free boson at the self-dual radius.

6.1. The defect operators $\hat{D}$ and $\hat{\bar{D}}$

The two operators $\hat{D}$ and $\hat{\bar{D}}$ associated with a defect D are related in a simple manner. Let $\{\varphi_i^{(1)}\}$ be a basis of $\mathcal{H}_1$ and $\{\varphi_j^{(2)}\}$ a basis of $\mathcal{H}_2$. Let the two-point functions on the sphere in CFT_1 and CFT_2 be given by

$$\langle\varphi_i^{(a)}(z)\varphi_j^{(a)}(w)\rangle = G_{ij}^{(a)}(z-w)^{-2h_i}(z^*-w^*)^{-2\bar{h}_i} \qquad \text{for} \quad a=1,2, \tag{6.1}$$

respectively. Note that $G_{ij}^{(a)}$ is related to the OPE coefficient $C_{ij}^{(a)\mathbf{1}}$ via $G_{ij}^{(a)} = C_{ij}^{(a)\mathbf{1}}\langle\mathbf{1}^{(a)}\rangle$, where $\mathbf{1}^{(a)}$ is the identity field of CFT_a.

Consider now a two-point correlator on the sphere where on one hemisphere we have CFT_1 with an insertion of $\varphi_i^{(1)}(z)$, while the other hemisphere supports CFT_2 with an insertion $\varphi_j^{(2)}(w)$, separated by the topological defect D. We may then deform the defect into a tight circle around either $\varphi_i^{(1)}$ or $\varphi_j^{(2)}$; this results in the identity

$$\langle(\hat{D}\varphi_i^{(1)})(z)\varphi_j^{(2)}(w)\rangle = \langle\varphi_i^{(1)}(z)(\hat{\bar{D}}\varphi_j^{(2)})(w)\rangle. \tag{6.2}$$

In terms of matrix elements, i.e. writing $\hat{D}\varphi_i^{(1)} = \sum_k d_{ik}\varphi_k^{(2)}$ and $\hat{\bar{D}}\varphi_j^{(2)} = \sum_l \overline{d}_{jl}\varphi_l^{(1)}$, this becomes

$$\sum_k G_{kj}^{(2)} d_{ik} = \sum_l G_{il}^{(1)}\overline{d}_{jl}. \tag{6.3}$$

Since $G^{(1)}$ and $G^{(2)}$ are invertible, this fixes $\overline{d}$ uniquely in terms of d.

The relation between $\hat{D}$ and $\hat{\bar{D}}$ is further simplified if the operator $\hat{D} : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible and preserves the two-point function on the sphere in the sense that

$$\langle(\hat{D}\varphi)(z)(\hat{D}\varphi')(w)\rangle = \xi\,\langle\varphi(z)\varphi'(w)\rangle \qquad \text{for all} \quad \varphi,\varphi' \in \mathcal{H}_1 \tag{6.4}$$

for some $\xi \in \mathbb{C}$. In this situation one has $\hat{\bar{D}} = \xi\,\hat{D}^{-1}$.

6.2. Defects preserving the $\widehat{u}(1)$-symmetry

Let us start by considering topological defects D within the free boson at a given radius R that obey (4.1) with $\epsilon = \bar{\epsilon} = 1$. In other words, D is transparent to the currents J and $\bar{J}$. This implies that the disorder fields starting the defect D carry a representation of the left and right copies of the $\widehat{u}(1)$-symmetry. Let θ be a $\widehat{u}(1)$-primary disorder field starting D of left and right $u(1)$-charges $(q,\bar{q})$ and suppose that $(q,\bar{q}) \in \Lambda(R)$. Let w be a point on the defect D such that there is no defect field insertion on D between w and the insertion point z of θ. Then there exists a $\widehat{u}(1)$-primary disorder field μ starting D with charges $(0, 0)$ such that

$$\theta(z) = \phi_{(q,\bar{q})}(z)\mu(w) \tag{6.5}$$

inside every correlator.

In words equation (6.5) states that the defect D can be detached from the disorder field $\theta(z)$, leaving a bulk field of the same charge at the point z. The new end point w of the defect is marked by a disorder field $\mu(w)$. Since μ has charge zero it obeys $L_{-1}\mu = 0 = \bar{L}_{-1}\mu$ and as a consequence correlators do not depend on the insertion point w.

The validity of (6.5) can be established as follows. Assuming for simplicity that $z = 0$, we consider the product of two bulk fields with $\theta(0)$. Taking the operator product of $\phi_{(-q,-\bar{q})}$ and θ, we have

$$\phi_{(q,\bar{q})}(u)\phi_{(-q,-\bar{q})}(v)\theta(0) = \phi_{(q,\bar{q})}(u)\sum_{l,r=0}^{\infty} v^{-q^2+l}(v^*)^{-\bar{q}^2+r}M_{l,r}\eta(0), \tag{6.6}$$

where $\eta(0)$ is a disorder field of charge $(0, 0)$ and $M_{r,l}$ denotes the appropriate combination of $\widehat{u}(1)$-modes of total left/right weight (l, r). The crucial point is now that

$$M_{l,r}\eta(0) = M_{l,r}\mathbf{1}(0)\eta(w). \tag{6.7}$$

To see this we observe that any combination of $\widehat{u}(1)$-modes in $M_{l,r}$ can be obtained as a suitable contour integral of the currents J and $\bar{J}$. Furthermore, by assumption the field η does not have any poles with these currents, and hence it may be moved through the contours at no cost. Finally, since $L_{-1}\eta = 0 = \bar{L}_{-1}\eta$, the correlator does not actually depend on the precise location of η, and hence we may move η from 0 to w. The modes $M_{l,r}$ now act on the identity field at 0. Since the field η does not create a branch cut for the currents J and $\bar{J}$, we can apply the same contour arguments for correlation functions involving η as for those involving the identity field $\mathbf{1}$. Thus, the operator product of two bulk fields results in the same combinations $M_{l,r}$ of modes as in the product in (6.6),

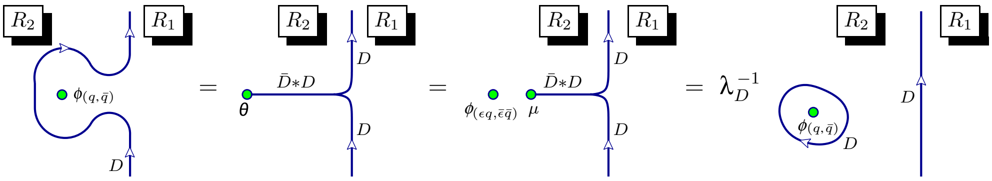
$$\phi_{(-q,-\bar{q})}(v)\phi_{(q,\bar{q})}(0) = C_{-q,q}\sum_{l,r=0}^{\infty} v^{-q^2+l}(v^*)^{-\bar{q}^2+r}M_{l,r}\phi_{(0,0)}(0), \tag{6.8}$$

where the structure constant $C_{-q,q}$ is the sign factor given in (3.5). Combining (6.6), (6.7) and (6.8) we can write

$$\phi_{(q,\bar{q})}(u)\phi_{(-q,-\bar{q})}(v)\theta(0) = \phi_{(q,\bar{q})}(u)\phi_{(-q,-\bar{q})}(v)\phi_{(0,0)}(0)((C_{-q,q})^{-1}\eta(w)). \tag{6.9}$$

Finally, applying the limit $\lim_{u\to v}|u-v|^{q^2}(\cdots)$ to both sides of this equality amounts to the replacement of $\phi_{(q,\bar{q})}(u)\phi_{(-q,-\bar{q})}(v)$ by the identity field. Setting $\mu = (C_{-q,q})^{-1}\eta$ then results in (6.5).

Let now D be a topological defect joining the theories $\mathrm{Bos}(R_1)$ and $\mathrm{Bos}(R_2)$, and obeying (4.1) for arbitrary ϵ and $\bar{\epsilon}$. Suppose further that D has the properties stated in Q3 in section 4.5, in particular that there is just a single defect field of left/right weight $(0, 0)$ living on D. Let $\phi_{(q,\bar{q})}(z)$ be a $\widehat{u}(1)$-primary bulk field of $\mathrm{Bos}(R_1)$ such that $(\epsilon q, \bar{\epsilon}\bar{q})$ lies in the charge lattice of $\mathrm{Bos}(R_2)$, i.e. $(q, \bar{q}) \in \Lambda(R_1) \cap \Lambda^{\epsilon,\bar{\epsilon}}(R_2)$. Then



(6.10)

for some nonzero constant λ_D. To see this, first fuse the part of D surrounding $\phi_{(q,\bar{q})}(z)$, which results in a defect line $\bar{D} * D$ ending on a disorder field $\theta(z)$. Since $\bar{D} * D$ obeys (4.1) with $\epsilon = \bar{\epsilon} = 1$ and since $\theta(z)$ has charges $(\epsilon q, \bar{\epsilon}\bar{q}) \in \Lambda(R_2)$ we can apply the relation (6.5). This results in an insertion of $\phi_{(\epsilon q,\bar{\epsilon}\bar{q})}(z)$ and of a weight zero disorder field $\mu(w)$. But a disorder field starting $\bar{D} * D$ is the same as a defect field living on D, and by assumption every weight zero defect field on D is proportional to the identity field on D. Thus we can write $\mu = a_{q,\bar{q}}\mathbf{1}_D$ for some constant $a_{q,\bar{q}} \in \mathbb{C}$. Altogether we hence obtain the equality

$$D\phi_{(q,\bar{q})}(z) = a_{q,\bar{q}}\phi_{(\epsilon q,\bar{\epsilon}\bar{q})}(z)D, \tag{6.11}$$

where the position of the symbol D indicates on which side of the defect line the bulk field is inserted. Closing the contour of D in (6.11) to a loop then results in the equality

$$\hat{D}\phi_{(q,\bar{q})} = \lambda_D a_{q,\bar{q}} \phi_{\epsilon q,\bar{\epsilon}\bar{q}}, \tag{6.12}$$

where the constant λ_D is determined by the action of $\hat{D}$ on the identity field,

$$\hat{D}\mathbf{1}^{(R_1)} = \lambda_D \mathbf{1}^{(R_2)}. \tag{6.13}$$

From (6.12) we see that λ_D has to be nonzero, or else the defect operator would vanish identically, contradicting the assumptions in Q3. Substituting (6.12) into (6.11) finally gives the last equality in (6.10).

We would now like to use (6.10) to determine the operator $\hat{D}$ as accurately as possible. As already noted in section 4.1, $\hat{D}$ is necessarily of the form

$$\hat{D} = \sum_{(q,\bar{q})\in\Lambda} d(q,\bar{q}) P(R_2, R_1)^{\epsilon q,\bar{\epsilon}\bar{q}}_{q,\bar{q}}, \tag{6.14}$$

where the maps $P(R_2, R_1)^{\epsilon q,\bar{\epsilon}\bar{q}}_{q,\bar{q}}$ are those that we denoted by $P^{\epsilon q,\bar{\epsilon}\bar{q}}_{q,\bar{q}}$ in (4.7). The operator for the action of $\bar{D}$ can be obtained from (6.3) to be

$$\hat{\bar{D}} = \frac{\langle \mathbf{1}^{(R_2)} \rangle}{\langle \mathbf{1}^{(R_1)} \rangle} \sum_{(q,\bar{q})\in\Lambda} d(-\epsilon q, -\bar{\epsilon}\bar{q}) P(R_1, R_2)^{\epsilon q,\bar{\epsilon}\bar{q}}_{q,\bar{q}}. \tag{6.15}$$

Recall the definition of $\hat{R}_2$ in (4.3). We will investigate the case that $\hat{R}_2/R_1$ is rational, so that Λ contains an infinite number of points. As in section 4.1.1 we set $\hat{R}_2/R_1 = M/N$ for coprime positive integers M and N, and note that the lattice Λ is spanned by the vectors $e_1 = N/(2R_1) \cdot (1, 1)$ and $e_2 = M R_1 \cdot (1, -1)$.

We select two vectors $(p, \bar{p}) = a e_1 + b e_2$ and $(q, \bar{q}) = c e_1 + d e_2$ in Λ and consider two insertions $\phi_{(p,\bar{p})}(z)$ and $\phi_{(q,\bar{q})}(0)$ of bulk fields of $\mathrm{Bos}(R_1)$ surrounded by a loop of defect D. Invoking the identity (6.10) we conclude that

$$\hat{D}\big(\phi_{(p,\bar{p})}(z)\phi_{(q,\bar{q})}(0)\big) = \lambda_D^{-1} (\hat{D}\phi_{(p,\bar{p})})(z)(\hat{D}\phi_{(q,\bar{q})})(0). \tag{6.16}$$

On the left-hand side of this equality we can take the operator product of the theory $\mathrm{Bos}(R_1)$, while on the right-hand side we use the operator product of $\mathrm{Bos}(R_2)$. Taking care of the sign factors in the OPE (3.5), in terms of the coefficients $d(\cdot,\cdot)$ in the decomposition (6.14) the condition reads

$$d(p+q, \bar{p}+\bar{q}) = \lambda_D^{-1} (\epsilon\bar{\epsilon})^{(ad-bc)MN} d(p,\bar{p}) d(q,\bar{q}). \tag{6.17}$$

Every solution to (6.17) is of the form

$$d(q,\bar{q}) = \lambda_D (\epsilon\bar{\epsilon})^{\frac{1}{2}(q^2-\bar{q}^2)} \alpha^q \beta^{\bar{q}} \tag{6.18}$$

for some $\alpha, \beta \in \mathbb{C}^\times$. To determine the constant λ_D we compute the torus partition function with an insertion of the defect loop $\bar{D} * D$ via a trace and use the modular transformation $\tau \mapsto -1/\tau$ to deduce the spectrum of defect operators on D:

$$\begin{aligned} &\mathrm{tr}_{\mathcal{H}(R_1)}\big(\hat{\bar{D}}\hat{D}\mathrm{q}^{\mathrm{L}_0-\frac{1}{24}}(\mathrm{q}^*)^{\bar{\mathrm{L}}_0-\frac{1}{24}}\big) \\ &\quad = \frac{\langle \mathbf{1}^{(R_2)} \rangle}{\langle \mathbf{1}^{(R_1)} \rangle} \lambda_D^2 \sum_{(p,\bar{p})\in\Lambda} \frac{\mathrm{q}^{\frac{1}{2}p^2}(\mathrm{q}^*)^{\frac{1}{2}\bar{p}^2}}{\eta(\mathrm{q})\eta(\mathrm{q}^*)} = \frac{\langle \mathbf{1}^{(R_2)} \rangle}{\langle \mathbf{1}^{(R_1)} \rangle} \frac{\lambda_D^2}{MN} \sum_{(p,\bar{p})\in\Lambda^*} \frac{\tilde{\mathrm{q}}^{\frac{1}{2}p^2}(\tilde{\mathrm{q}}^*)^{\frac{1}{2}\bar{p}^2}}{\eta(\tilde{\mathrm{q}})\eta(\tilde{\mathrm{q}}^*)}. \end{aligned} \tag{6.19}$$

Here Λ^* is the lattice dual to Λ, while $\mathrm{q} = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\tilde{\mathrm{q}} = \mathrm{e}^{2\pi\mathrm{i}(-1/\tau)}$ give the dependence on the modular parameter of the torus. For $\hat{D}$ to be fundamental, the multiplicity space of weight zero fields should be one dimensional, so that we need $\lambda_D^2 = MN\langle \mathbf{1}^{(R_1)} \rangle / \langle \mathbf{1}^{(R_2)} \rangle$. It remains to determine the sign of λ_D. This is done in appendix D. One finds a remaining ambiguity which amounts to choosing, once and for all, a square root of $\langle \mathbf{1}^{(R)}\rangle$ for each $R > 0$. To summarize the findings so far, the fundamental defect operators from $\mathcal{H}(R_1)$ to $\mathcal{H}(R_2)$ and obeying (4.1) are given by

$$\hat{D}(x, y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1} = \sqrt{\frac{\langle \mathbf{1}^{(R_1)}\rangle MN}{\langle \mathbf{1}^{(R_2)}\rangle}} \sum_{(q,\bar{q})\in\Lambda} (\epsilon\bar{\epsilon})^{\frac{1}{2}(q^2-\bar{q}^2)}\, \mathrm{e}^{2\pi \mathrm{i}(xq-y\bar{q})}\, P(R_2, R_1)^{\epsilon q,\bar{\epsilon}\bar{q}}_{q,\bar{q}}. \tag{6.20}$$

The form given in (4.6) is then obtained by choosing $\sqrt{\langle \mathbf{1}^{(R)}\rangle} \equiv 1$.

6.3. Virasoro-preserving defects

Let us start with the free boson compactified at the self-dual radius. To arrive at the formula stated in (4.18) for the fundamental defect operators preserving only the Virasoro symmetry we could proceed analogously to [15], which uses factorization of bulk two-point correlators on the upper half plane as in [18]. In terms of the decomposition (4.16) of $\mathcal{H}(R_{\mathrm{s.d.}})$, this amounts to picking Virasoro-primaries $\phi, \tilde{\phi} \in \mathcal{H}_{[s,\bar{s}]}$ and $\psi, \tilde{\psi} \in \mathcal{H}_{[t,\bar{t}]}$, and considering the complex plane with a defect D running along the real axis and bulk field insertions of $\phi(z)$, $\tilde{\phi}(z^*)$, $\psi(w)$ and $\tilde{\psi}(w^*)$. Comparing the limits $\mathrm{Im}(z), \mathrm{Im}(w) \to 0$ and $|z - w| \to 0$ one finds constraints on the defect operator $\hat{D}$, which can be solved in terms of representation matrices of $SL(2, \mathbb{C})$. However, when describing the defect just as a boundary condition in the folded system, not all ways of analysing topological defects can be applied. This is for instance the case for the method used below, since it relies on the deformation and fusion of defect lines.

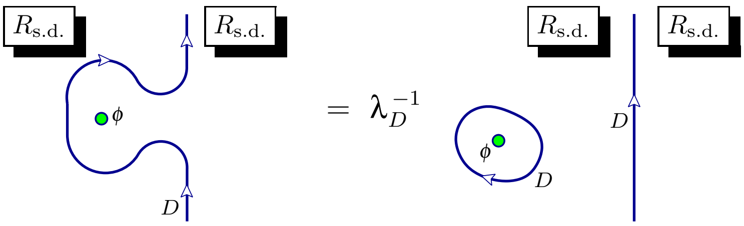
We start by investigating the properties of topological defects X that can end on a disorder field μ which is Virasoro-primary of weights $(h, \bar{h}) = \left(\frac{1}{4}, \frac{1}{4}\right)$. To this end, we consider the monodromies of the $su(2)$ currents J^a and $\bar{J}^a$ around μ. By factorization it is enough to investigate the monodromy of the two three-point correlators

$$f(\tilde{\mu}; z) = \langle J^a(z)\mu(0)\tilde{\mu}(-L)\rangle \qquad \text{and} \qquad g(\tilde{\mu}; z) = \langle \bar{J}^a(z)\mu(0)\tilde{\mu}(-L)\rangle \tag{6.21}$$

on the complex plane with a defect X stretched from 0 to $-L$, and $\tilde{\mu}$ an arbitrary disorder field that can terminate an X-defect. We will show that $f(\tilde{\mu}; z)$ and $g(\tilde{\mu}; z)$ are single-valued on $\mathbb{C}\backslash\{0, -L\}$ for all choices of $\tilde{\mu}$. It is enough to consider $\tilde{\mu}$ that are Virasoro-primary, since going to descendents does not affect $f(\tilde{\mu}; z)$ and $g(\tilde{\mu}; z)$ being single-valued or not. Next note that $J^a \in \mathcal{H}_{[2,0]}$ and so the fusion rules dictated by the Virasoro null vectors imply that $f(\tilde{\mu}; z)$ can be nonzero only if $\tilde{\mu}$ has conformal weights $(h, \bar{h}) = (s^2/4, \bar{s}^2/4)$ with $(s, \bar{s}) = (1, 1)$ or $(s, \bar{s}) = (3, 1)$. Thus $f(\tilde{\mu}; z)$ is proportional to $z^{-1}(z + L)^{-1}L^{\frac{1}{2}}$ for $(s, \bar{s}) = (1, 1)$, proportional to $z(z + L)^{-3}L^{-\frac{3}{2}}$ for $(s, \bar{s}) = (3, 1)$ and zero otherwise. In particular $f(\tilde{\mu}; z)$ is single-valued for all choices of $\tilde{\mu}$. A similar argument shows that $g(\tilde{\mu}; z)$ is single-valued. It follows that the $su(2)$ currents are single-valued close to $\mu(0)$.

Suppose now that X is of the form $\bar{D} * D$ for some defect D obeying the properties stated in Q3. The positivity assumption made there implies that the disorder field μ can be written as a sum of disorder fields which are $(J_0, \bar{J}_0)$-eigenstates. To see this let U be the space of disorder fields starting the defect X that are Virasoro-primary of weight $\left(\frac{1}{4}, \frac{1}{4}\right)$. By positivity, these fields are annihilated by the positive $\hat{u}(1)$-modes, and the action of L_0 and $\bar{L}_0$ is thus given by $L_0 = \frac{1}{2}J_0J_0$ and $\bar{L}_0 = \frac{1}{2}\bar{J}_0\bar{J}_0$, respectively. Since $L_0, \bar{L}_0$ and $J_0, \bar{J}_0$ commute, the modes J_0 and $\bar{J}_0$ map the $L_0, \bar{L}_0$-eigenspace U to itself. We can always bring J_0 and $\bar{J}_0$ into Jordan normal form, and since the action of L_0 and $\bar{L}_0$ is diagonal with nonzero eigenvalue, it follows that J_0 and $\bar{J}_0$ must also be diagonalizable. We conclude that μ is a sum of disorder fields with $u(1)$-charges $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, all of which are contained in the charge lattice $\Lambda(R_{\mathrm{s.d.}})$. Defects ending on the disorder field μ therefore behave as those associated with

$(q, \bar{q}) \in \Lambda(R)$ in the previous subsection and we can thus use the same arguments which lead to (6.10) to deduce that for a primary bulk field $\phi \in \mathcal{H}_{[1,1]}$ we have



(6.22)

where D is an arbitrary defect with the properties stated in Q3 (not necessarily one that can end on μ as above).

One can check by recursion that every Virasoro-primary bulk field appears in the repeated fusion of primary fields in the sector $\mathcal{H}_{[1,1]}$, cf [15]. We can use the same recursion to show that (6.22) applies to all bulk fields. Indeed, suppose (6.22) holds for $\psi \in \mathcal{H}_{[s,\bar{s}]}$. Let us write (6.22) symbolically as $D\phi(z) = \lambda_D^{-1}(\hat{D}\phi)(z)D$. Then

$$D\phi(z)\psi(w) = \lambda_D^{-2}(\hat{D}\phi)(z)(\hat{D}\psi)(w)D \tag{6.23}$$

for $\phi \in \mathcal{H}_{[1,1]}$. Taking the OPE on both sides and comparing terms implies that (6.22) also holds for the primary fields in $\mathcal{H}_{[s\pm1,\bar{s}\pm1]}$.

The defect operator $\hat{D}$ obeys (2.2) and thus maps the sector $\mathcal{H}_{[s,\bar{s}]}$ to itself. It follows from (6.23) with the same arguments as in section 6.2 that $\lambda_D^{-1}\hat{D}$ has to be a homomorphism of the bulk OPE. Since the bulk fields are generated by the elements of $\mathcal{H}_{[1,1]}$, the operator $\hat{D}$ is uniquely determined by its action on $\mathcal{H}_{[1,1]}$.

The restriction of $\hat{D}$ to $\mathcal{H}_{[1,1]}$ can be written as

$$\hat{D}\big|_{\mathcal{H}_{[1,1]}} = \lambda_D R \otimes \mathrm{id}_{\mathcal{H}^{\mathrm{Vir}}_{1/4} \otimes \bar{\mathcal{H}}^{\mathrm{Vir}}_{1/4}} \qquad \text{with} \quad R \in \mathrm{End}(V_{1/2} \otimes \overline{V}_{1/2}). \tag{6.24}$$

Consistency with the bulk OPE poses constraints on the linear map R. These are analysed in appendix E, with the result that we can always find $g, h \in SL(2, \mathbb{C})$ such that

$$R = g \otimes h. \tag{6.25}$$

The restriction of $\hat{D}$ to $\mathcal{H}_{[1,1]}$ defines $\hat{D}$ uniquely, and thus we could in principle construct the full defect operator inductively from (6.24). It is, however, simpler to obtain this operator in a different manner. To this end, we consider the family of topological defects that is obtained by perturbing the trivial defect by the bulk field

$$\phi = aJ^+ + bJ^3 + cJ^- + \tilde{a}\bar{J}^+ + \tilde{b}\bar{J}^3 + \tilde{c}\bar{J}^-. \tag{6.26}$$

This field is clearly self-local and thus generates a truly marginal deformation (see section 5.5). The corresponding defect operator is simply the exponential of the corresponding zero modes as given in (4.17),

$$\hat{D} = \exp\left(aJ_0^+ + bJ_0^3 + cJ_0^- + \tilde{a}\bar{J}_0^+ + \tilde{b}\bar{J}_0^3 + \tilde{c}\bar{J}_0^-\right). \tag{6.27}$$

By construction, J_0^a acts on the $V_{s/2} \otimes V_{\bar{s}/2}$ part of $\mathcal{H}_{[s,\bar{s}]}$ via the representation $R_{s/2}(J^a) \otimes \mathrm{id}$ of the Lie algebra $sl(2, \mathbb{C})$, and similarly $\bar{J}_0^a$ acts as $\mathrm{id} \otimes R_{\bar{s}/2}(J^a)$. By exponentiating we obtain the representations $\rho_{s/2}$ and $\rho_{\bar{s}/2}$ of the Lie group, so that altogether we arrive at the operator (4.18). It is then immediate to deduce the rule for orientation reversal as $\bar{D}(g, h) = D(g^{-1}, h^{-1})$. In particular, all defect operators $\hat{D}(g, h)$ are fundamental, since the torus amplitude with an insertion of $\bar{D} * D$ is equal to the torus amplitude without defects, which contains the vacuum with multiplicity one.

The operators $\hat{D}(g,h)$ in fact constitute all fundamental Virasoro-preserving defect operators for $\mathrm{Bos}(R_{\mathrm{s.d.}})$. This follows since the restriction of $\hat{D}(g,h)$ to $\mathcal{H}_{[1,1]}$ is $g\otimes h\otimes \mathrm{id}$. Combining this with the previous result (6.25) we see that for any given fundamental $\hat{D}$ we can find g,h such that $\hat{D}=\lambda_D\hat{D}(g,h)$. The amplitude of a torus with insertion of $\bar{D}*D$ is then equal to $\lambda_D^2 Z(R_{\mathrm{s.d.}})$. Using the assumptions in Q3, we conclude $\lambda_D=\pm 1$. Finally, $\lambda_D=-1$ would lead to all coefficients in the torus amplitude with an insertion of a singe D-defect being negative. Hence $\lambda_D=1$.

Since the set (4.20) gives all fundamental defect operators we can recover the $\widehat{u}(1)$-preserving operators for $R_1=R_2=R_{\mathrm{s.d.}}$. Note that $(\sqrt{2}R_2)^{\epsilon\bar{\epsilon}}/\sqrt{2}=R_{\mathrm{s.d.}}$, so that we are in the case treated in section 4.1.1, with $\hat{R}_2/R_1=1$. It is enough to compare the action on the subspace $\mathcal{H}_{[1,1]}$. The Virasoro-primaries in this space are just the $\widehat{u}(1)$-primary fields $\phi_{(\pm 1/\sqrt{2},\pm 1/\sqrt{2})}$. In the representation $V_{\frac{1}{2}}$ of $SL(2,\mathbb{C})$ we choose the standard basis $e_1=\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle$ and $e_2=\left|j=\frac{1}{2},m=-\frac{1}{2}\right\rangle$. Fixing in addition an appropriate vector $\varphi\in\mathcal{H}^{\mathrm{Vir}}_{s^2/4}\otimes\bar{\mathcal{H}}^{\mathrm{Vir}}_{\bar{s}^2/4}$, we can write[11]

$$\begin{aligned} e_1\otimes\bar{e}_1\otimes\varphi &= \phi_{(1/\sqrt{2},1/\sqrt{2})}, & e_1\otimes\bar{e}_2\otimes\varphi &= \phi_{(1/\sqrt{2},-1/\sqrt{2})}, \\ e_2\otimes\bar{e}_1\otimes\varphi &= \phi_{(-1/\sqrt{2},1/\sqrt{2})}, & e_2\otimes\bar{e}_2\otimes\varphi &= -\phi_{(-1/\sqrt{2},-1/\sqrt{2})}. \end{aligned} \tag{6.28}$$

Comparing (4.6) and (4.18) then establishes (4.25), as well as, e.g.,

$$D(g,h)=D(x,y)^{+,-}_{R_{\mathrm{s.d.}},R_{\mathrm{s.d.}}},$$
$$g=\begin{pmatrix}\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}\sqrt{2}x} & 0\\ 0 & -\mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}\sqrt{2}x}\end{pmatrix},\qquad h=\begin{pmatrix}0 & -\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}\sqrt{2}y}\\ -\mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}\sqrt{2}y} & 0\end{pmatrix}. \tag{6.29}$$

Up to now we have concentrated on $\mathrm{Bos}(R_{\mathrm{s.d.}})$. Let us now also give some details on how to arrive at the parametrization (4.24) for the defect operators (4.22) defined at rational multiples of $R_{\mathrm{s.d.}}$. On a vector $|j,m\rangle$ in the spin-j representation V_j of $\mathrm{sl}(2,\mathbb{C})$, Γ_L acts as

$$\rho(\Gamma_L)|j,m\rangle=\mathrm{e}^{2\pi\mathrm{i}m/L}|j,m\rangle \tag{6.30}$$

($j\in\mathbb{Z}/2, m+j\in\mathbb{Z}, |m|\leqslant j$). Let us denote a basis of Virasoro-highest weight states in the sector $\mathcal{H}_{[s,\bar{s}]}$ (cf the decomposition (4.16) as $\left|\frac{s}{2},m;\frac{\bar{s}}{2},\bar{m}\right\rangle$. Selecting the maximal torus of $SU(2)$ that consists of diagonal matrices as the one corresponding to the $u(1)$-charge, the state $\left|\frac{s}{2},m;\frac{\bar{s}}{2},\bar{m}\right\rangle$ has $u(1)$-charges $\sqrt{2}(m,\bar{m})$. This state is in the image of $\hat{D}(0,0)^{+,+}_{R_{\mathrm{s.d.}},R_1}$ iff $\sqrt{2}(m,\bar{m})\in\Lambda(R_1)$. Comparing with the expression (4.15) for Λ we see that this is the case if and only if $(m,\bar{m})=\frac{1}{2}(kF_1+lE_1,kF_1-lE_1)$ for some $k,l\in\mathbb{Z}$. Altogether, we therefore have

$$\sqrt{2}(m,\bar{m})\in\Lambda(R_1)\quad\Leftrightarrow\quad m+\bar{m}\in F_1\mathbb{Z}\quad\text{and}\quad m-\bar{m}\in E_1\mathbb{Z}. \tag{6.31}$$

This implies that for $\left|\frac{s}{2},m;\frac{\bar{s}}{2},\bar{m}\right\rangle$ in the image of $\hat{D}(0,0)^{+,+}_{R_{\mathrm{s.d.}},R_1}$ (these are the only states on which $\hat{D}(g,h)$ in (4.22) actually acts) we have

$$\hat{D}\big(\Gamma_{E_1},\Gamma_{E_1}^{-1}\big)\left|\frac{s}{2},m;\frac{\bar{s}}{2},\bar{m}\right\rangle=\mathrm{e}^{2\pi\mathrm{i}(m-\bar{m})/E_1}\left|\frac{s}{2},m;\frac{\bar{s}}{2},\bar{m}\right\rangle=\left|\frac{s}{2},m;\frac{\bar{s}}{2},\bar{m}\right\rangle. \tag{6.32}$$

The same holds for $\hat{D}\big(\Gamma_{F_1},\Gamma_{F_1}\big)$, so that we conclude that

$$\begin{aligned}\hat{D}\big(\Gamma^k_{E_1},\Gamma^{-k}_{E_1}\big)\circ\hat{D}(0,0)^{+,+}_{R_{\mathrm{s.d.}},R_1}&=\hat{D}(0,0)^{+,+}_{R_{\mathrm{s.d.}},R_1},\\ \hat{D}\big(\Gamma^k_{F_1},\Gamma^{k}_{F_1}\big)\circ\hat{D}(0,0)^{+,+}_{R_{\mathrm{s.d.}},R_1}&=\hat{D}(0,0)^{+,+}_{R_{\mathrm{s.d.}},R_1}\end{aligned} \tag{6.33}$$

[11] The minus sign in the last term comes from (3.5). To see this note that $e_2=J_0^-e_1$ and $\bar{e}_2=\bar{J}_0^-\bar{e}_1$, where we chose $J^-(z)=\phi_{(-\sqrt{2},0)}(z)$ and $\bar{J}^-(z)=\phi_{(0,-\sqrt{2})}(z)$.

for all $k \in \mathbb{Z}$. In the same way, one can show that

$$\begin{aligned} \hat{D}(0,0)^{+,+}_{R_2,R_{\mathrm{s.d.}}} \circ \hat{D}\big(\Gamma^k_{E_2}, \Gamma^{-k}_{E_2}\big) &= \hat{D}(0,0)^{+,+}_{R_2,R_{\mathrm{s.d.}}}, \\ \hat{D}(0,0)^{+,+}_{R_2,R_{\mathrm{s.d.}}} \circ \hat{D}\big(\Gamma^k_{F_2}, \Gamma^{k}_{F_2}\big) &= \hat{D}(0,0)^{+,+}_{R_2,R_{\mathrm{s.d.}}} \end{aligned} \tag{6.34}$$

for all $k \in \mathbb{Z}$. Combining these observations with the composition law (4.19) for the defect operators at the self-dual radius, we see that $\hat{D}(g,h)_{R_2,R_1} = \hat{D}(g',h')_{R_2,R_1}$ if

$$(g',h') = \big(\Gamma^{k_2}_{E_2}\Gamma^{l_2}_{F_2} g \Gamma^{k_1}_{E_1}\Gamma^{l_1}_{F_1}, \Gamma^{-k_2}_{E_2}\Gamma^{l_2}_{F_2} h \Gamma^{-k_1}_{E_1}\Gamma^{l_1}_{F_1}\big) \tag{6.35}$$

for some $k_1, k_2, l_1, l_2 \in \mathbb{Z}$. One can convince oneself that these identifications together with $(g',h') = (-g,-h)$ are the only cases for which $\hat{D}(g,h)_{R_2,R_1} = \hat{D}(g',h')_{R_2,R_1}$, so that we arrive at formula (4.24).

One can also argue, similarly as in [16], that the defect operators (4.22) together with the $\widehat{u}(1)$-preserving defect operators (6.20) describe already all fundamental defect operators (in the sense of question Q3 of section 4.5). In particular, one can obtain the individual projectors onto the Virasoro-irreducible sectors which occur in both $\mathcal{H}(R_1)$ and $\mathcal{H}(R_2)$ as appropriate integrals over these defect operators. Any additional defect operator can therefore be written as a sum or integral over these operators, and hence is not fundamental.

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Appendix A. A family of defects with common defect operator

Here, we present an example of a marginal perturbation of a defect D by a non-selfadjoint defect field that leads to a family of distinct defects $D(\lambda)$ which nonetheless all have the same defect operator $\hat{D}(\lambda) = \hat{D}$. In particular, the defects $D(\lambda)$ can then not be distinguished in correlators involving only defect lines and bulk fields, but no disorder or defect fields.

For the free boson theory $\mathrm{Bos}(R)$, let the defect D be a superposition of the trivial defect and a $\widehat{u}(1)$-preserving defect as investigated in section 4.1.1,

$$D = D_0 + D_1 \qquad \text{with} \quad D_0 = D(0,0)^{+,+}_{R,R}, \qquad D_1 = D(\sqrt{2},0)^{+,+}_{R,R}. \tag{A.1}$$

The spectrum of defect-changing fields that change D_0 to D_1 (when passing along the defect in the direction of its orientation) is the same as the spectrum of disorder fields that start the defect D_1; it can be computed as in (6.19). We find

$$\mathrm{tr}_{\mathcal{H}(R)}\big(\hat{D}_1 \mathrm{q}^{L_0-\frac{1}{24}} (\mathrm{q}^*)^{\overline{L}_0-\frac{1}{24}}\big) = \sum_{(p,\bar{p})\in\Lambda(R)} \frac{\tilde{\mathrm{q}}^{\frac{1}{2}(p+\sqrt{2})^2} (\tilde{\mathrm{q}}^*)^{\frac{1}{2}\bar{p}^2}}{\eta(\tilde{\mathrm{q}})\eta(\tilde{\mathrm{q}}^*)}. \tag{A.2}$$

Let us for simplicity assume that $R > 2R_{\mathrm{s.d.}}$. Then the field of lowest conformal weight is the $\widehat{u}(1)$-primary defect-changing field with $u(1)$-charges $(q,\bar{q}) = (\sqrt{2},0)$ and conformal weights $(h,\bar{h}) = (1,0)$; we denote this field by θ. This is one of the fields that according to the results of section 5.5 is truly marginal[12]. Denote by $\mathbf{1}_0$ the identity field on the defect D_0

[12] While in section 5.5 only $R^2 \in \mathbb{Q}$ was considered, θ is truly marginal for all values of R. For example, it has regular (in fact, vanishing) operator product with itself.

(which is the same as the identity field in the bulk) and by $\mathbf{1}_1$ the identity field on D_1. Consider a patch of local coordinates on the world sheet where the defect D runs along the real axis. Then by construction, for $x > y > z$ we have

$$\mathbf{1}_1(x)\theta(y) = \theta(y) = \theta(y)\mathbf{1}_0(z), \qquad \mathbf{1}_0(x)\theta(y) = 0 = \theta(y)\mathbf{1}_1(z). \tag{A.3}$$

In particular, combining these relations with the associativity of the OPE we see that θ has vanishing operator product with itself,

$$\theta(x)\theta(z) = \theta(x)\mathbf{1}_1(y)\theta(z) = 0. \tag{A.4}$$

Let now $D(\lambda)$ be the defect obtained by perturbing D by $\lambda\theta$ for some $\lambda \in \mathbb{C}$. Correlators involving $D(\lambda)$ are obtained from correlators involving D by inserting the operator $\exp(\lambda \int_D \theta(x)\mathrm{d}x)$. It is then easy to see that $\hat{D}(\lambda) = \hat{D}$: consider a loop of defect D; expanding the exponential gives

$$\mathbf{1}_D + \lambda \int_D \theta(x)\,\mathrm{d}x + \frac{1}{2}\lambda^2 \int_D \int_D \theta(x)\theta(y)\,\mathrm{d}x\,\mathrm{d}y + \cdots. \tag{A.5}$$

In this expansion all terms with more than one insertion of θ vanish owing to (A.4). The term with one θ-insertion vanishes because we can replace $\theta(x)$ by $\theta(x)\mathbf{1}_0$ and then drag the field $\mathbf{1}_0$ around the loop so as to arrive at the product $\mathbf{1}_0\theta(x)$, which is zero.

One can now ask whether the defects $D(\lambda)$ are at all different from $D = D(0)$. There are at least two ways to see that this is indeed the case. The first is to note that $D(\lambda)$, while still transparent to $T(z)$, is no longer transparent to $J(z)$ if $\lambda \neq 0$. The second way is to compute the Hamiltonian generating translations along a cylinder with $D(\lambda)$ running along the Euclidean time direction; this Hamiltonian turns out to be non-diagonalizable for $\lambda \neq 0$.

Let us elaborate on the second issue. Consider the cylinder obtained by the identification $w \sim w + 2\pi\mathrm{i}$ on the complex plane, with the defect $D(\lambda)$ put on the (equivalence class of the) real axis. This geometry can be mapped to the full complex plane with coordinate z by $z = \mathrm{e}^w$. In the z-coordinates, the defect $D(\lambda)$ runs along the positive real axis; the Hamiltonian then takes the form

$$H(\lambda) = L_0 + \bar{L}_0 - \tfrac{1}{12} + \lambda\theta(1). \tag{A.6}$$

The Hamiltonian $H(\lambda)$ acts on the space $\mathcal{H}_D$ of disorder fields which create the defect $D(\lambda)$. To expose the non-trivial Jordan cell structure of $H(\lambda)$, consider the three vectors

$$\begin{aligned}
|a_0\rangle &= |0\rangle - \lambda \int_0^1 \mathrm{d}x\,\theta(x)|0\rangle, \\
|a_1\rangle &= \theta(0)|0\rangle, \\
|b_1\rangle &= J_{-1}|0\rangle - \lambda J_{-1} \int_0^1 \mathrm{d}x\,\theta(x)|0\rangle + \lambda\sqrt{2} \int_0^1 \mathrm{d}x\,x^{-1}(\theta(x) - \theta(0))|0\rangle.
\end{aligned} \tag{A.7}$$

First note that the integrals are finite. For $\lambda = 0$ the three vectors form a basis of the $(L_0, \bar{L}_0)$-eigenspaces of $\mathcal{H}_D$ with eigenvalues $(0, 0)$ and $(1, 0)$. Using $[J_m, \theta(x)] = \sqrt{2}x^m\theta(x)$ and $[L_0 + \bar{L}_0, \theta(x)] = \frac{\partial}{\partial x}(x\theta(x))$, one verifies that

$$H(\lambda)|a_0\rangle = -\tfrac{1}{12}|a_0\rangle, \quad H(\lambda)|a_1\rangle = \left(1 - \tfrac{1}{12}\right)|a_1\rangle, \quad H(\lambda)|b_1\rangle = \left(1 - \tfrac{1}{12}\right)|b_1\rangle - \lambda\sqrt{2}|a_1\rangle. \tag{A.8}$$

Thus, the operator $H(\lambda)$ acts on the vectors $\{|a_0\rangle, |a_1\rangle, |b_1\rangle\}$ as the 3×3 matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\lambda\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{12}\mathbf{1}_{3\times 3}, \tag{A.9}$$

which for $\lambda \neq 0$ contains a non-trivial Jordan block.

Appendix B. Representation theory of algebras in $\mathcal{U}_N$

In this appendix, we present details of the calculations referred to in section 5. We assume some familiarity with the methods of [7, 28, 11].

B.1. The modular tensor category $\mathcal{U}_N$

To define the braided tensor category $\mathcal{U}_N$ we describe the tensor product and braiding in a basis. Our conventions are given in [7, section 2.2]. The index set for the simple objects U_k of $\mathcal{U}_N$ is $\mathcal{I} = \{0, 1, \dot{s}, 2N-1\}$. The fusion rules are $U_k \otimes U_\ell \cong U_{[k+\ell]}$, where $k, \ell \in \mathcal{I}$ and we set

$$[k+\ell] := k+\ell \bmod 2N \in \mathcal{I}. \tag{B.1}$$

The dual of $k \in \mathcal{I}$ is $\overline{k} = 2N - k$, and all simple objects have unit quantum dimension, $\dim(U_k) = 1$. The relevant fusing matrices can be found in [35] or [7, section 2.5.1]. With a suitable choice of bases in the morphism spaces $\mathrm{Hom}(U_i \otimes U_j, U_k)$, which for $\mathcal{U}_N$ are all one dimensional, the twists θ, the fusing matrices F and the braiding matrices R are given by

$$\theta_k = \mathrm{e}^{-\pi \mathrm{i} k^2/(2N)}, \qquad \mathsf{R}^{(k\ell)[k+\ell]} = \mathrm{e}^{-\pi \mathrm{i} k\ell/(2N)}, \qquad \mathsf{F}^{(rst)[r+s+t]}_{[s+t][r+s]} = (-1)^{r\sigma(s+t)}, \tag{B.2}$$

where $k, \ell, r, s, t \in \mathcal{I}$, and $\sigma(k+\ell)$ is 0 if $k+\ell < 2N$ and 1 otherwise. We denote the bases of $\mathrm{Hom}(U_i \otimes U_j, U_k)$ for which these equalities hold by $\lambda_{(i,j)k}$.

From these data the s-matrix (related to the modular S-matrix by $s_{i,j} = S_{i,j}/S_{0,0}$) is found to be

$$s_{k,\ell} = \mathrm{e}^{-\pi \mathrm{i} k\ell/N}. \tag{B.3}$$

B.2. Frobenius algebras in $\mathcal{U}_N$

Every haploid special symmetric Frobenius algebra in $\mathcal{U}_N$ is isomorphic to one of the algebras A_r defined as follows [28, section 3.3]. As an object in $\mathcal{U}_N$ we have

$$A_r = \bigoplus_{a=0}^{r-1} U_{2aN/r} \qquad \text{where} \qquad r \in \mathbb{Z}_{>0} \text{ divides } N. \tag{B.4}$$

For $a \in \mathbb{Z}_r$ we denote by $e_a \in \mathrm{Hom}(U_{2aN/r}, A_r)$ and $r_a \in \mathrm{Hom}(A_r, U_{2aN/r})$ embedding and restriction morphisms for the subobject $U_{2aN/r}$ of A_r (hence $r_a \circ e_a = \mathrm{id}_{U_{2aN/r}}$). One can choose the multiplication and unit morphism of A_r to be

$$m = \sum_{a,b=0}^{r-1} e_{a+b} \circ \lambda_{(2aN/r,2bN/r)[2(a+b)N/r]} \circ (r_a \otimes r_b) \qquad \text{and} \qquad \eta = e_0. \tag{B.5}$$

With the help of the fusing matrices in (B.2) it is straightforward to verify associativity of m and the unit property. The isomorphism class of A_r is specified by the Kreuzer–Schellekens bihomomorphism (KSB) $\Xi^{(r)}$, which is given by the scalars $\Xi^{(r)}(b, a)$ that are defined by the equality [28, section 3.4]

$$m \circ c_{A_r,A_r} \circ (e_a \otimes e_b) = \Xi^{(r)}(b, a) m \circ (e_a \otimes e_b), \tag{B.6}$$

where c_{A_r,A_r} is the self-braiding of A_r. For the multiplication stated in (B.5) this gives

$$\Xi^{(r)}(b, a) = \mathrm{e}^{-2\pi \mathrm{i} Nab/r^2}. \tag{B.7}$$

B.3. A_r-bimodules in $\mathcal{U}_N$

According to [11, proposition 5.16], simple A_r-bimodules in $\mathcal{U}_N$ are labelled by elements of the Abelian group $G^{(r,N)} = H^* \times_H \mathrm{Pic}(\mathcal{U}_N)$, where $\mathrm{Pic}(\mathcal{U}_N) = \mathbb{Z}_{2N}$ and $H = \mathbb{Z}_r$ is embedded in $\mathrm{Pic}(\mathcal{U}_N)$ via $\iota(a) = 2aN/r$. Accordingly, $h \in H$ acts on $k \in \mathrm{Pic}(\mathcal{U}_N)$ by $h.k = k + \iota(h)$; the action of h on $\psi \in H^*$ is given by $(h.\psi)(a) = \psi(a)\Xi^{(r)}(a, h)$.

If we label the elements of H^* by $x \in \mathbb{Z}_r$ via

$$\psi_x(a) = \mathrm{e}^{-2\pi \mathrm{i} xa/r} \tag{B.8}$$

then with the value (B.7) of the KSB we find

$$(h.\psi_x)(a) = \mathrm{e}^{-2\pi \mathrm{i} xa/r}\, \mathrm{e}^{-2\pi \mathrm{i} Nha/r^2} = \psi_{x+hN/r}(a) \tag{B.9}$$

for $h \in \mathbb{Z}_r$. The action of H on $\mathrm{Pic}(\mathcal{U}_N)$ is simply $h.k = k + 2hN/r$. Thus, the group $G^{(r,N)}$ is given by $G^{(r,N)} = \mathbb{Z}_r \times_{\mathbb{Z}_r} \mathbb{Z}_{2N}$ where the product over $\mathbb{Z}_r$ means that $(a, k{+}2N/r) = (a{+}N/r, k)$. Denote by t_ψ the automorphism of the algebra A_r associated with $\psi \in H^*$, see [11, section 5.1]. A simple bimodule corresponding to an element $(\psi, k) \in G^{(r,N)}$ is provided by $\alpha^+_{A_r}(U_k)_{t_\psi}$, i.e. an alpha-induced bimodule for which the right action of A_r is twisted by the automorphism t_ψ, see [11, section 5.2] for definitions. By [11, proposition 5.16], the fusion of two such bimodules is given by addition in $G^{(r,N)}$.

It is convenient to label the A_r-bimodules by elements of the group $G_{n,P,Q}$ defined in (5.5), where $N = n^2PQ$ and $r = nP$. The map

$$\varphi : \quad (\alpha, \beta, \gamma) \mapsto (\alpha - \beta, 2\beta + \gamma) \tag{B.10}$$

from $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$ induces a surjective group homomorphism from $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ to $G^{(r,N)}$ whose kernel is given by the subgroup generated by the elements $(r, 0, 0)$, $(0, N/r, 0)$ and $(1, 1, -2)$. By comparison with the definition of $G_{n,P,Q}$ this means that φ induces a group isomorphism

$$\hat{\varphi} : \quad G_{n,P,Q} \xrightarrow{\cong} G^{(r,N)}. \tag{B.11}$$

Next we compute the action of the corresponding defects on bulk fields. Bulk fields are labelled by bimodule morphisms $\phi \in \mathrm{Hom}_{A_r,A_r}(U_x \otimes^+ A_r \otimes^- U_y, A_r)$ (see [11, section 2] for definitions). The action of the defect D_B corresponding to the bimodule $B = \alpha^+_{A_r}(U_k)_{t_{\psi_a}}$ is defined in [11, equation (2.30)], and an expression in terms of quantities in $\mathcal{U}_N$ can be found in [11, equation (4.14)]. A short calculation, which uses in particular various properties of the algebra A_r and the fact that ϕ is a morphism of bimodules in order to get rid of the A_r-loop, shows that

$$\hat{D}_B(\phi) = \frac{s_{x,k}}{s_{x,0}} t_{\psi_a} \circ \phi \circ \left(\mathrm{id}_{U_x} \otimes t^{-1}_{\psi_a} \otimes \mathrm{id}_{U_y}\right) = \mathrm{e}^{-\pi \mathrm{i}((k+a)x+ay)/N} \phi. \tag{B.12}$$

If, as in section 5.3.1, we use an element $(a, b, \rho) \in G_{n,P,Q}$ to label the defect, then according to the isomorphism (B.11) we have

$$\hat{D}^{(r)}_{(a,b,\rho)}(\phi) = \mathrm{e}^{-\pi \mathrm{i}(a(x+y)+b(x-y)+\rho x)/N} \phi. \tag{B.13}$$

Conversely, substituting $q = x/\sqrt{2N}$, $\bar{q} = y/\sqrt{2N}$, $R = \sqrt{P/(2Q)}$ and $N = n^2PQ$ into (5.8), one recovers formula (B.13).

B.4. A_r–A_s-bimodules in $\mathcal{U}_N$

For r and s divisors of N the algebra A_ℓ, with $\ell = \mathrm{lcm}(r, s)$, has A_r and A_s as subalgebras. As in section 5.3.2, let $A^{(rs)}$ be the A_r–A_s-bimodule obtained by considering A_ℓ as an A_ℓ–A_ℓ-bimodule over itself and restricting the left action to A_r and the right action to A_s. We abbreviate $X = A^{(rs)}$.

Let $\psi_x^{(r)} \in (\mathbb{Z}_r)^*$ be defined as in (B.8). Given two characters $\psi_x^{(r)} \in (\mathbb{Z}_r)^*$ and $\psi_y^{(s)} \in (\mathbb{Z}_s)^*$ we set $X_{x,y} = {}_{t_x}X_{t_y}$, with $t_x = t_{\psi_x^{(r)}}$ and $t_y = t_{\psi_y^{(s)}}$. Thus $X_{x,y}$ is the bimodule obtained from X by twisting the left and right actions by the algebra automorphisms defined by the characters $\psi_x^{(r)}$ and $\psi_y^{(s)}$, see [11, section 5.1].

We would like to compute the space $H_{x,y}{:=}\mathrm{Hom}_{A_r,A_s}(X, X_{x,y})$ of bimodule intertwiners. Since $X \cong A_\ell$ as objects of $\mathcal{U}_N$, an element $f \in H_{x,y}$ can be expanded as

$$f = \sum_{a=0}^{\ell-1} f_a e_a \circ r_a, \tag{B.14}$$

where $f_a \in \mathbb{C}$ and e_a and r_a are the embedding and restriction morphisms for $U_{2aN/\ell}$ as a subobject of A_ℓ, as introduced in appendix B.2. Denote by e_{A_r} the embedding of A_r into A_ℓ, by e_{A_s} that of A_s into A_ℓ and by m the multiplication of A_ℓ. Then the condition for f to be a bimodule intertwiner is

$$f \circ m \circ (m \otimes \mathrm{id}_{A_\ell}) \circ (e_{A_r} \otimes \mathrm{id}_{A_\ell} \otimes e_{A_s}) = m \circ (m \otimes \mathrm{id}_{A_\ell}) \circ ((e_{A_r} \circ t_x) \otimes f \otimes (e_{A_s} \circ t_y)). \tag{B.15}$$

Evaluating this equality for appropriate choices of simple subobjects, one finds that it holds if and only if

$$f_{a\ell/r+b\ell/s+d} = \psi_x^{(r)}(a)\psi_y^{(s)}(b) f_d \qquad \text{for all} \quad a \in \mathbb{Z}_r, \quad b \in \mathbb{Z}_s, \quad d \in \mathbb{Z}_\ell. \tag{B.16}$$

The space of solutions of this equality is one dimensional if $x - y \in g\,\mathbb{Z}$, with $g = \gcd(r, s)$, and zero dimensional otherwise, i.e. $\dim_{\mathbb{C}}(H_{x,y}) = \delta_{x,y}^{[g]}$. In particular, X is simple. Similarly one finds that $X_{x,y} \cong X_{u,v}$ if and only if $x - u \equiv y - v \bmod g$.

The object $A_r \otimes A_s$ is an A_r–A_s-bimodule in the obvious way. It is easy to check that the morphism

$$\left(t_x^{-1} \otimes t_y^{-1}\right) \circ \left(r_{A_r} \otimes r_{A_s}\right) \circ \Delta, \tag{B.17}$$

where r_{A_r} and r_{A_s} are the restriction morphisms corresponding to e_{A_r} and e_{A_s} and Δ is the coproduct of A_ℓ, constitutes a nonzero intertwiner of bimodules from $X_{x,y}$ to $A_r \otimes A_s$. Moreover, for $x = 0, 1, \ldots, g-1$ the bimodules $X_{x,0}$ are mutually non-isomorphic, and hence we can decompose $A_r \otimes A_s$ as a bimodule as

$$A_r \otimes A_s \cong \bigoplus_{x=0}^{g-1} X_{x,0} \oplus Y \tag{B.18}$$

for some A_r–A_s-bimodule Y. Computing the quantum dimensions on both sides one finds $rs = g\ell + \dim(Y)$, which implies $\dim(Y) = 0$ and hence $Y = 0$.

Since every simple A_r–A_s-bimodule B is a sub-bimodule of $A_r \otimes U_k \otimes A_s$ for some k, the decomposition (B.18) implies that it is also a sub-bimodule of $U_k \otimes^+ X_{x,0}$ and of $X_{x,0} \otimes^+ U_k$ for some k and x (the same holds with $\otimes^-$). It is straightforward to see that this implies the isomorphisms (5.10), e.g. $X_{x,0} \otimes^+ U_k = X_{x,0} \otimes_{A_s} \alpha^+_{A_s}(U_k) = X_{0,0} \otimes_{A_s} \alpha^+_{A_s}(U_k)_{t'_x} = A^{(rs)} \otimes_{A_s} B^{(s)}_{a,b,\rho}$.

Let us finally sketch the computation of the action of the defect corresponding to X on bulk fields. Fix a basis $\phi_{x,y}^{(r)} \in \mathrm{Hom}_{A_r,A_r}(U_x \otimes^+ A_r \otimes^- U_y, A_r)$ by setting

$$\phi_{x,y}^{(r)} \circ \left(\mathrm{id}_{U_x} \otimes \eta \otimes \mathrm{id}_{U_y}\right) = e^{(r)}_{[x+y]r/(2N)} \circ \lambda_{(x,y)[x+y]} \tag{B.19}$$

where $e_a^{(r)} \in \mathrm{Hom}(U_{2aN/r}, A_r)$. (This differs from the basis chosen in section 3 by phases.) Since the morphism spaces $\mathrm{Hom}_{A_r,A_r}(U_x \otimes^+ A_r \otimes^- U_y, A_r)$ are either zero or one dimensional, we have $\hat{D}_X \phi^{(s)}_{x,y} = \xi \phi^{(r)}_{x,y}$ for some $\xi \in \mathbb{C}$. To determine ξ we evaluate relation (4.14) of [11] (with $X_\nu = X$, $X_\mu = A = A_r$, and $B = A_s$). After a while we obtain

$$\xi = \delta^{[2N/r]}_{x+y,0}\delta^{[2N/s]}_{x+y,0}\delta^{[2s]}_{x-y,0}\frac{1}{r}\sum_{m=0}^{\ell-1}\exp\left(-2\pi\mathrm{i}\frac{m}{\ell}\frac{x-y}{2}\right). \tag{B.20}$$

This gives rise to the relation (5.11) (recall that $\ell = \mathrm{lcm}\,(r,s)$).

B.5. The A_r-bimodule D and self-locality

Consider the free boson compactified at radius $R = 2^{-1/2}E/F$ (i.e. $P = E^2$ and $Q = F^2$) in terms of the algebra A_{nE^2} in $\mathcal{U}_{(nEF)^2}$. Set $a = nEF$ and $r = nE^2$.

Let us first check that every defect field θ of conformal weight $(h,\bar{h}) = (1,0)$ which is also a J_0-eigenvector is self-local. Such a θ is an element of the sector $U_k \otimes \overline{U}_0$, for $k = 0$, $k = 2a$ or $k = 2N - 2a$. The important point to note is that according to (B.2) the braiding is trivial,

$$c_{U_k,U_k} = \mathrm{id}_{U_k\otimes U_k} \qquad \text{for} \quad k = 0, 2a, 2N-2a. \tag{B.21}$$

Using the TFT formalism it is then straightforward to verify the identity (5.17) (see e.g. [11, section 4.1] for the TFT representation of some defect correlators). The case that θ has weight (0, 1) can be treated analogously.

Next let us compute the spectrum of defect fields on the defect D defined in (5.18). Denote the bimodule labelling the defect D by B, and recall that according to (B.10) the bimodule labelling the elementary defect $D^{(r)}_{(c,d,\rho)}$ is the twisted alpha-induced bimodule $\alpha^+_{A_r}(U_{2d+\rho})_{t_{c-d}}$, where again $t_x \equiv t_{\psi^{(r)}_x}$. Thus we have the decomposition

$$B = \bigoplus_{k=0}^{E-1}\bigoplus_{l=0}^{F-1}\alpha^+_{A_r}(U_{2al})_{t_{a(k-l)}}. \tag{B.22}$$

Defect fields on D in the sector $U_i \otimes \overline{U}_j$ are labelled by elements of the morphism space $H_{ij} = \mathrm{Hom}_{A_r,A_r}(U_i \otimes^+ B \otimes^- U_j, B)$, so that the multiplicity Z^D_{ij} in formula (5.19) is given by $Z^D_{ij} = \dim_{\mathbb{C}}(H_{ij})$. Using (B.22), H_{ij} can be written as a direct sum of spaces of the form

$$\mathrm{Hom}_{A_r,A_r}\big(U_i \otimes^+ \alpha^+_{A_r}(U_k)_{t_x} \otimes^- U_j, \alpha^+_{A_r}(U_\ell)_{t_y}\big) \cong \mathrm{Hom}_{A_r,A_r}(X, A_r) \tag{B.23}$$

with

$$X = \big(U_i \otimes^+ \alpha^+_{A_r}(U_k)_{t_x} \otimes^- U_j\big) \otimes_{A_r} \big(\alpha^+_{A_r}(U_\ell)_{t_y}\big)^\vee \cong \alpha^+_{A_r}(U_{[i+j+k-\ell]})_{\chi^{-1}_{U_j} t_x t_y}, \tag{B.24}$$

where the second isomorphism, with $\chi_{U_j}(2bN/r) = S_{j,2bN/r}/S_{j,0} = \mathrm{e}^{-\pi\mathrm{i}jb/r}$, holds owing to [11, propositions 5.8 and 5.9]. Putting these results together, we find

$$Z^D_{ij} = \sum_{k,k'=0}^{E-1}\sum_{l,l'=0}^{F-1} d^{a(k-l),a(k'-l')}_{2la,2l'a} \tag{B.25}$$

with $d^{x,y}_{k,\ell} = \dim_{\mathbb{C}}(\mathrm{Hom}_{A_r,A_r}(X, A_r))$. Using in addition [11, proposition 5.10] and (B.7) we can write $d^{x,y}_{k,\ell}$ as

$$d^{x,y}_{k,\ell} = \delta^{[2N/r]}_{i+j+k-\ell,0}\delta_{\Xi_{A_r}(\cdot,(i+j+k-\ell)r/(2N))t_x t_y,\chi_{U_j}} = \delta^{[2N/r]}_{i+j+k-\ell,0}\delta^{[2r]}_{i-j+k-\ell+2(x-y),0}. \tag{B.26}$$

After a short calculation one then indeed obtains formula (5.19) (recall that $N = a^2$).

What remains to complete the demonstrations of the claims in section 5.5 is to give a maximal self-local subspace of defect fields on the defect D. As a first step we construct a basis for the spaces

$$L_x := \mathrm{Hom}_{A_r,A_r}(U_{2ax} \otimes^+ B, B) \qquad \text{and} \qquad R_x := \mathrm{Hom}_{A_r,A_r}(B \otimes^- U_{2ax}, B) \tag{B.27}$$

with $x \in \{0, 1, a-1\}$ (note that $a^2 = N$). Denote by $e_{k,l}$ and $r_{k,l}$ the bimodule intertwiners furnishing the embedding and restriction morphisms for the simple sub-bimodule $\alpha^+_{A_r}(U_{2al})_{t_{a(k-l)}}$ of B. Let e_a and r_a be as in (B.5) and denote by $\bar\lambda^{(i,j)k}$ the basis vector in $\mathrm{Hom}(U_k, U_i \otimes U_j)$ dual to $\lambda_{(i,j)k}$ in the sense that $\lambda_{(i,j)k} \circ \bar\lambda^{(i,j)k} = \mathrm{id}_{U_k}$. Consider the morphisms

$$\begin{aligned}
\alpha(x)^{k',l'}_{k,l} := {} & e_{k',l'} \circ \big(m \otimes \mathrm{id}_{U_{2al'}}\big) \circ \big(\mathrm{id}_{A_r} \otimes e_{2a(l-l'+x)r/N} \otimes \mathrm{id}_{U_{2al'}}\big) \\
& \circ \big(\mathrm{id}_{A_r} \otimes \big(\bar\lambda^{([2a(l-l'+x)],2al')[2a(l+x)]} \circ \lambda_{(2ax,2al)[2a(l+x)]}\big)\big) \\
& \circ \big(c_{U_{2ax},A_r} \otimes \mathrm{id}_{U_{2al}}\big) \circ \big(\mathrm{id}_{U_{2ax}} \otimes r_{k,l}\big) \in \mathrm{Hom}(U_{2ax} \otimes B, B), \\
\beta(x)^{k',l'}_{k,l} := {} & e_{k',l'} \circ \big(m \otimes \mathrm{id}_{U_{2al'}}\big) \circ \big(\mathrm{id}_{A_r} \otimes e_{2a(l-l'+x)r/N} \otimes \mathrm{id}_{U_{2al'}}\big) \\
& \circ \big(\mathrm{id}_{A_r} \otimes \big(\bar\lambda^{([2a(l-l'+x)],2al')[2a(l+x)]} \circ \lambda_{(2al,2ax)[2a(l+x)]}\big)\big) \\
& \circ \big(r_{k,l} \otimes \mathrm{id}_{U_{2ax}}\big) \in \mathrm{Hom}(B \otimes U_{2ax}, B).
\end{aligned} \tag{B.28}$$

These are nonzero if and only if $U_{2a(l-l'+x)}$ is a subobject of A_r, i.e. iff $a(l-l'+x)r/N \in \mathbb{Z}$. Furthermore it is easy to check that these morphisms intertwine the left action of A_r. For the right action a small calculation is needed, the conclusion being that $\alpha(x)^{k',l'}_{k,l}$ intertwines the right action iff $k' \equiv k+x \bmod E$, and that $\beta(x)^{k',l'}_{k,l}$ intertwines the right action iff $k' \equiv k-x \bmod E$. For $u \in \mathbb{Z}$ let $[u]^K$ be the element of $\{0, 1, \ldots, K-1\}$ that is equal to u modulo K. (The relation to the bracket notation $[\cdot]$ as introduced in (B.1) is thus $[u] \equiv [u]^{2N}$.) Then altogether we have

$$\alpha(x)_{k,l} \equiv \alpha(x)^{[k+x]^E,[l+x]^F}_{k,l} \in L_x \qquad \text{and} \qquad \beta(x)_{k,l} \equiv \beta(x)^{[k-x]^E,[l+x]^F}_{k,l} \in R_x \tag{B.29}$$

for $k \in \{0, 1, \ldots, E-1\}$ and $l \in \{0, 1, \ldots, F-1\}$. The morphisms $\alpha(x)_{k,l}$ and $\beta(x)_{k,l}$ are all nonzero. It is also easy to verify that they are linearly independent (compose a zero linear combination with $e_{k,l}$ from the right to isolate the individual terms). In fact, $\alpha(x)_{k,l}$ and $\beta(x)_{k,l}$ provide bases of L_x and R_x, respectively, as can be checked directly or by using that by (5.19) $\dim(L_x) = \dim(R_x) = EF$.

Using the TFT representation of correlators, one can verify that for defect fields labelled by

$$\theta \in \mathrm{Hom}_{A_r,A_r}(U_{2ax} \otimes^+ B \otimes^- U_{2ay}, B) \quad \text{and} \quad \theta' \in \mathrm{Hom}_{A_r,A_r}(U_{2ax'} \otimes^+ B \otimes^- U_{2ay'}, B), \tag{B.30}$$

with $x, y, x', y' \in \{0, 1, a-1\}$ such that at least one of x, y is zero and at least one of x', y' is zero, the self-locality condition (5.17) is equivalent to

$$\theta \circ \big(\mathrm{id}_{U_{2ax}} \otimes \theta' \otimes \mathrm{id}_{U_{2ay}}\big) \circ \big(c^{-1}_{U_{2ax},U_{2ax'}} \otimes \mathrm{id}_B \otimes c_{U_{2ay},U_{2ay'}}\big) = \theta' \circ \big(\mathrm{id}_{U_{2ax'}} \otimes \theta \otimes \mathrm{id}_{U_{2ay'}}\big). \tag{B.31}$$

Let us analyse this condition using the bases (B.29). For a collection $t = \{t_{k,l} | k = 0, 1, \ldots, E-1, l = 0, 1, \ldots, F-1\}$ of EF numbers we set

$$\theta[x,t]_L := \sum_{k=0}^{E-1} \sum_{l=0}^{F-1} t_{k,l}\, \alpha(x)_{k,l} \qquad \text{and} \qquad \theta[x,t]_R := \sum_{k=0}^{E-1} \sum_{l=0}^{F-1} t_{k,l}\, \beta(x)_{k,l}. \tag{B.32}$$

Set now $\theta = \theta[x, t]_L$ and $\theta' = \theta[x', t']_L$ in (B.31). To evaluate the resulting condition on t and t' compose both sides with $e_{k,l}$ to remove the summation. Then rewrite the morphisms on either side using the F and R matrices as given in (B.2). This step is simplified by the fact that

$$\mathsf{F}^{(2e,2f,2g)[2e+2f+2g]}_{[2f+2g][2e+2f]} = 1 \qquad \text{and} \qquad \mathsf{R}^{(2ae,2af)[2a(e+f)]} = 1 \tag{B.33}$$

for all e, f, g. One finds that (B.31) holds if and only if

$$t_{[k+x']^E,[l+x']^F} \cdot t'_{k,l} = t_{k,l} \cdot t'_{[k+x]^E,[l+x]^F} \tag{B.34}$$

for all k, l. Similar conditions result when setting $\theta = \theta[x, t]_L, \theta' = \theta[x', t']_R$ and $\theta = \theta[x, t]_R, \theta' = \theta[x', t']_R$ in (B.31). All these conditions are fulfilled if $t_{k,l}$ and $t'_{k,l}$ are independent of k and l. We therefore introduce the vector space

$$\mathcal{L} := \mathrm{span}_{\mathbb{C}}\{\theta[0, t]_L, \theta[1, t]_L, \theta[a-1, t]_L, \theta[0, t]_R, \theta[1, t]_R, \theta[a-1, t]_R \text{ with } t_{k,l} \equiv 1\}. \tag{B.35}$$

$\mathcal{L}$ is a six-dimensional self-local subspace of $L_0 \oplus L_1 \oplus L_{a-1} \oplus R_0 \oplus R_1 \oplus R_{a-1}$.

Let now $\theta[x, t]_L \in L_x$ be arbitrary and suppose that the space $\mathrm{span}_{\mathbb{C}}\{\mathcal{L}, \theta[x, t]_L\}$ is self-local. Then in particular (B.34) has to hold for $t'_{k,l} \equiv 1$ and $x' = 1$, i.e., for all k, l we have $t_{k,l} = t_{[k+1]^E,[l+1]^F}$. This implies $t_{k,l} = t_{[k-l]^E,0}$ and $t_{k,0} = t_{[k+mF]^E,0}$ for all $m \in \mathbb{Z}$. Since E and F are coprime, the latter condition leads to $t_{k,0} = t_{0,0}$ and therefore $t_{k,l} = t_{0,0}$ for all k, l. Thus already $\theta[x, t]_L \in \mathcal{L}$. A similar argument applies to $\theta[x, t]_R$. Thus $\mathcal{L}$ is maximal.

Appendix C. Morita equivalence of $\tilde{A}_1$ and $\tilde{A}_N$ in $\mathcal{D}_N$

Let us first describe the fusion rules in the modular tensor category $\mathcal{D}_N$, which can be extracted from [36]. The category $\mathcal{D}_N$ has $N + 7$ isomorphism classes of simple objects. We denote a choice of representatives by

$$\{\mathbf{1}, J, V_1, V_2, \ldots, V_{N-1}, W_0, W_1\} \cup \{\sigma_0, \sigma_1, \tau_0, \tau_1\}. \tag{C.1}$$

We also use the notation $J_\alpha, \sigma_\alpha, \tau_\alpha$ for $\alpha \in \mathbb{Z}_2$ and V_r for $r = 0, 1, \ldots, N$, where we set

$$J_0 = \mathbf{1}, \qquad J_1 = J, \qquad V_0 = \mathbf{1} \oplus J, \qquad V_N = W_0 \oplus W_1. \tag{C.2}$$

The twist eigenvalues (given by $\mathrm{e}^{-2\pi \mathrm{i}\Delta}$ with Δ the conformal weight) and the quantum dimensions are

$$\begin{array}{c|ccccc} & J_\alpha & W_\alpha & V_r & \sigma_\alpha & \tau_\alpha \\ \hline \theta & 1 & \mathrm{i}^{-N} & \mathrm{e}^{-\pi \mathrm{i} r^2/(2N)} & \mathrm{e}^{-\pi \mathrm{i}/8} & \mathrm{e}^{-\pi \mathrm{i} 9/8} \\ \dim & 1 & 1 & 2 & \sqrt{N} & \sqrt{N} \end{array} \tag{C.3}$$

for $\alpha \in \mathbb{Z}_2$ and $r = 0, 1, \ldots, N$. For $u \in \{0, 1, \ldots, 2N\}$ we set

$$\{u\} := \begin{cases} u & \text{if} \quad u \leqslant N, \\ 2N - u & \text{if} \quad u > N. \end{cases} \tag{C.4}$$

The fusion rules in the untwisted sector are

$$\begin{aligned} &V_r \otimes V_s \cong V_{\{r+s\}} \oplus V_{|r-s|}, && J_\alpha \otimes V_r \cong V_r, && W_\alpha \otimes V_r \cong V_{N-r}, \\ &J_\alpha \otimes J_\beta \cong J_{\alpha+\beta}, && J_\alpha \otimes W_\beta \cong W_{\alpha+\beta}, && W_\alpha \otimes W_\beta \cong J_{\alpha+\beta+N} \end{aligned} \tag{C.5}$$

for $\alpha, \beta \in \mathbb{Z}_2$ and $r, s \in \{0, 1, \ldots, N\}$. The fusion rules involving fields from the twisted sector are given by

$$\begin{aligned} &J_\alpha \otimes \sigma_\beta \cong \delta^{[2]}_{\alpha,0}\sigma_\beta \oplus \delta^{[2]}_{\alpha,1}\tau_\beta, \qquad W_\alpha \otimes \sigma_\beta \cong \delta^{[2]}_{\alpha,\beta+N}\sigma_{\beta+N} \oplus \delta^{[2]}_{\alpha,\beta+N+1}\tau_{\beta+N}, \\ &V_r \otimes \sigma_\alpha \cong \sigma_{\alpha+r} \oplus \tau_{\alpha+r}, \\ &\sigma_\alpha \otimes \sigma_\beta \cong \delta^{[2]}_{\alpha,\beta} W_\alpha \oplus \delta^{[2]}_{\alpha+\beta+N,0}\left(\mathbf{1} \oplus \bigoplus_{k=1, k\,\text{even}}^{N-1} V_k\right) \oplus \delta^{[2]}_{\alpha+\beta+N,1}\left(\bigoplus_{k=1, k\,\text{odd}}^{N-1} V_k\right). \end{aligned} \tag{C.6}$$

By verifying which fusions contain the tensor unit **1**, one finds the duality to be

$$(V_r)^\vee \cong V_r, \quad (J_\alpha)^\vee \cong J_\alpha, \quad (W_\alpha)^\vee \cong W_{\alpha+N}, \quad (\sigma_\alpha)^\vee \cong \sigma_{\alpha+N}, \quad (\tau_\alpha)^\vee \cong \tau_{\alpha+N}. \tag{C.7}$$

Let us now consider the algebras $\tilde{A}_1$ and $\tilde{A}_N$ in $\mathcal{D}_N$. The algebra $\tilde{A}_1 = \mathbf{1} \oplus J$ corresponds to the decomposition of the chiral algebra $\widehat{u}(1)_N$ into representations of its subalgebra $\widehat{u}(1)_N/\mathbb{Z}_2$. Conversely, the category of local left modules of $\tilde{A}_1$ in $\mathcal{D}_N$ can be made into a modular tensor category $(\mathcal{D}_N)^{\ell oc}_{\tilde{A}_1}$ (see, e.g., [31, section 3.4] for details and references), which is in fact equivalent to $\mathcal{U}_N$. The free boson compactified at radius $R = 1/\sqrt{2N}$ can be described by either using the algebra A_1 with the chiral algebra $\widehat{u}(1)_N$ or using $\tilde{A}_1$ with $\widehat{u}(1)_N/\mathbb{Z}_2$.

Consider the simple induced $\tilde{A}_1$-module

$$\Sigma := \tilde{A}_1 \otimes \sigma_0 \tag{C.8}$$

(one can also use σ_1 in place of σ_0). The object $\Sigma^\vee \otimes_{\tilde{A}_1} \Sigma$ carries again a natural structure of simple symmetric Frobenius algebra [37, proposition 2.13]; we denote this algebra by $\tilde{A}_N$. The algebra $\tilde{A}_N$ constructed in this way is Morita equivalent to $\tilde{A}_1$ [37, theorem 2.14]. The $\tilde{A}_1$–$\tilde{A}_N$-bimodule X and $\tilde{A}_N$–$\tilde{A}_1$-bimodule X' which furnish the Morita-context are $X = \Sigma$ and $X' = \Sigma^\vee$. In particular,

$$\dim(X) = \dim(X') = \dim(\tilde{A}_1)\dim(\sigma_0) = 2\sqrt{N}. \tag{C.9}$$

Using the braiding to take $\tilde{A}_1$ past σ_0, one can turn $\tilde{A}_N$ into a left $\tilde{A}_1$-module (this can be done in two ways, it does not matter which of them one chooses). Using that $\tilde{A}_1$ is commutative and that $\sigma_0^\vee \otimes \sigma_0$ is transparent to $\tilde{A}_1$, it is not hard to check that this way $\tilde{A}_N$ becomes in fact a local $\tilde{A}_1$-module. Let us denote this local module by B_N. It turns out that the multiplication can be lifted as well, so that B_N becomes an algebra in the category $(\mathcal{D}_N)^{\ell oc}_{\tilde{A}_1}$. The object in $\mathcal{D}_N$ underlying B_N is

$$\tilde{A}_N \cong \tilde{A}_1 \otimes \sigma_0^\vee \otimes \sigma_0 \cong \sum_{m=0}^{N-1} V_{\{2m\}}. \tag{C.10}$$

This also shows that $\dim(\tilde{A}_N) = 2N$. Via the equivalence $(\mathcal{D}_N)^{\ell oc}_{\tilde{A}_1} \simeq \mathcal{U}_N$ the corresponding object in $\mathcal{U}_N$ is $\bigoplus_{m=0}^{N-1} U_{2m}$. Up to isomorphism, this object carries a unique structure of a special symmetric Frobenius algebra, namely A_N. The image of B_N under the equivalence is therefore isomorphic to A_N as a symmetric Frobenius algebra. Thus all correlators which can be described using the algebra A_N with the chiral algebra $\widehat{u}(1)_N$ can alternatively be described using $\tilde{A}_N$ and $\widehat{u}(1)_N/\mathbb{Z}_2$.

Appendix D. Sign of $\widehat{u}(1)$-preserving defect operators

Here we investigate the sign of the parameter λ_D which appears in formula (6.18). From evaluating the trace (6.19) we know that we can write the parameter λ_D of the defect operator $\hat{D}(x,y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1}$ as

$$\lambda_D = \sigma(x,y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1} \sqrt{MN}\sqrt{\langle \mathbf{1}^{(R_1)}\rangle}/\sqrt{\langle \mathbf{1}^{(R_2)}\rangle} \qquad \text{with} \qquad \sigma(x,y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1} \in \{\pm 1\}. \tag{D.1}$$

For this expression to be unambiguous, let us agree that $\sqrt{MN} > 0$ and let us choose, once and for all, a square root $\sqrt{\langle \mathbf{1}^{(R)}\rangle}$ for each value of R.

To determine the signs $\sigma(x,y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1}$ first note that if $R_1 = R_2 =: R$, then one can consider instead of (6.19) an analogous trace with the insertion of only a single defect operator

$\hat{D}(x, y)^{\epsilon,\bar{\epsilon}}_{R,R}$. This leads to an overall factor of λ_D; positivity of the coefficients in the dual channel then enforces

$$\sigma(x, y)^{\epsilon,\bar{\epsilon}}_{R,R} = 1. \tag{D.2}$$

Note that acting with $\hat{D}(x, y)^{\epsilon,\bar{\epsilon}}_{R,R}$ on the identity field $\mathbf{1}^{(R)}$ of Bos(R) results in $\sqrt{MN}\mathbf{1}^{(R)}$. In particular, the coefficient is positive.

Next consider two defects $D_1 = D(x, y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1}$ and $D_2 = D(u, v)^{\nu,\bar{\nu}}_{R_1,R_2}$. The fused defect $D = D_2 * D_1$ acts on the identity field of Bos(R_1) as

$$\hat{D}\mathbf{1}^{(R_1)} = \sigma(u, v)^{\nu,\bar{\nu}}_{R_1,R_2}\sigma(x, y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1} MN\mathbf{1}^{(R_1)}. \tag{D.3}$$

Since D is a $\widehat{u}(1)$-preserving defect of Bos(R_1), by the above result the coefficient appearing here is positive. This is possible for all choices of parameters only if $\sigma(x, y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1}$ is independent of x, y and $\epsilon, \bar{\epsilon}$. Thus

$$\sigma(x, y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1} = \sigma_{R_2,R_1} \qquad \text{and} \qquad \sigma_{R,R} = 1. \tag{D.4}$$

Consider the (rather big and non-connected) graph Γ obtained by taking a vertex for every positive real number R and a directed edge $e(R_1, R_2)$ between two vertices R_1 and R_2 if either R_1/R_2 or R_1R_2 is rational (thus for each edge $e(R_1, R_2)$ there is also an edge $e(R_2, R_1)$). To a directed edge $e(R_1, R_2)$ assign the value σ_{R_1,R_2}. Let γ be a closed path in Γ. Let $R_1, R_2, \ldots, R_{n+1} = R_1$ be the vertices traversed by the path. By the same argument as above, acting with the fused defect

$$D(x_n, y_n)^{\epsilon_n\bar{\epsilon}_n}_{R_1R_n} * \cdots * D(x_2, y_2)^{\epsilon_2\bar{\epsilon}_2}_{R_3R_2} * D(x_1, y_1)^{\epsilon_1\bar{\epsilon}_1}_{R_2R_1} \tag{D.5}$$

on $\mathbf{1}^{(R_1)}$ shows that $\sigma_{R_1R_n}\cdots\sigma_{R_3R_2}\sigma_{R_2R_1} = 1$. One can therefore write $\sigma_{R_2R_1} = \sigma_{R_1}\cdot\sigma_{R_2}$ for some function $\sigma : \mathbb{R}_{>0} \to \{\pm 1\}$. (In choosing σ one has the freedom of an over-all sign on each connected component of Γ.)

Comparing the result found so far, $\sigma(x, y)^{\epsilon,\bar{\epsilon}}_{R_2,R_1} = \sigma_{R_2}\sigma_{R_1}$ with (D.1) we see that the freedom that is left in determining the signs of the numbers λ_D is precisely the freedom to choose the square roots $\sqrt{\langle \mathbf{1}^{(R)}\rangle}$.

Appendix E. Constraints on Virasoro-preserving defects

In this appendix, we show that one can always choose the parametrization (6.25) of the linear map R in (6.24). Select a highest weight state $\varphi_{s,\bar{s}}$ in each of the spaces $\mathcal{H}^{\mathrm{Vir}}_{s^2/4} \otimes \bar{\mathcal{H}}^{\mathrm{Vir}}_{\bar{s}^2/4}$. The Virasoro-highest weight states in $\mathcal{H}_{[s,\bar{s}]}$ can then be written as $u \otimes \varphi_{s,\bar{s}}$ with $u \in V_{s/2} \otimes \overline{V}_{\bar{s}/2}$. Denote the corresponding primary field by $[u \otimes \varphi_{s,\bar{s}}](z)$. We identify $V_0 \otimes \overline{V}_0$ with $\mathbb{C}$ and take $\varphi_{0,0}$ to be the identity field; we also abbreviate $W = V_{1/2} \otimes \overline{V}_{1/2}$.

The leading terms in the OPE of two primary fields in $\mathcal{H}_{[1,1]}$ are of the form

$$\begin{aligned}[u \otimes \varphi_{1,1}](z)[v \otimes \varphi_{1,1}](w) &= A_{0,0}(u, v)r^{-1}\varphi_{0,0}(w) + \mathrm{e}^{\mathrm{i}\vartheta}[A_{2,0}(u, v) \otimes \varphi_{2,0}](w)\\ &\quad + \mathrm{e}^{-\mathrm{i}\vartheta}[A_{0,2}(u, v) \otimes \varphi_{0,2}](w) + O(r),\end{aligned} \tag{E.1}$$

where $r = |z - w|$, $\vartheta = \arg(z - w)$ and $A_{s,\bar{s}} : W \times W \to V_{s/2} \otimes \overline{V}_{\bar{s}/2}$ are bilinear maps. Compatibility with the action of the $\widehat{su}(2)$ zero modes requires $A_{s,\bar{s}}$ to be an intertwiner,

$$A_{s,\bar{s}}((g \otimes h)u, (g \otimes h)v) = (\rho_{s/2}(g) \otimes \rho_{s/2}(h))A_{s,\bar{s}}(u, v) \tag{E.2}$$

for all $g, h \in SL(2, \mathbb{C})$ and $u, v \in W$.

The defect operator $\hat{D}$ restricted to $\mathcal{H}_{s,\bar{s}}$ is of the form

$$\hat{D}|_{\mathcal{H}_{[s,\bar{s}]}} = \lambda_D R_{s,\bar{s}} \otimes \mathrm{id}_{\mathcal{H}^{\mathrm{Vir}}_{s^2/4} \otimes \bar{\mathcal{H}}^{\mathrm{Vir}}_{\bar{s}^2/4}} \qquad \text{with} \qquad R_{s,\bar{s}} \in \mathrm{End}(V_{s/2} \otimes \overline{V}_{\bar{s}/2}). \tag{E.3}$$

To match the notation in (6.24) we abbreviate $R_{1,1} =: R$. Applying the compatibility condition (6.23) between the action of the defect and the OPE of bulk fields to the OPE (E.1) yields the conditions

$$A_{s,\bar{s}}(Ru, Rv) = R_{s,\bar{s}} \circ A_{s,\bar{s}}(u, v) \qquad \text{for all} \quad u, v \in W. \tag{E.4}$$

on the linear maps $R_{s,\bar{s}}$.

Let us start by analysing this condition for $A_{0,0}$. First of all, non-degeneracy of the bulk two-point correlator implies that $A_{0,0}$ furnishes a non-degenerate pairing on $W \times W$. Furthermore, $A_{0,0}$ is invariant with respect to the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, and all such bilinear forms on $W \times W$ are symmetric. Denote by $O(W)$ the group of all endomorphisms of W that leave $A_{0,0}$ invariant.

From (6.22) it follows that $\hat{D}\varphi_{0,0} = \lambda_D \varphi_{0,0}$. This in turn implies $R_{0,0} = 1$. As a consequence, $A_{0,0}(Ru, Rv) = A_{0,0}(u, v)$, i.e. $R \in O(W)$. Since W is a complex vector space, and since $A_{0,0}$ is non-degenerate and symmetric, there is a basis $\{v_1, v_2, v_3, v_4\}$ such that $A_{0,0}(v_i, v_j) = \delta_{i,j}$. The isomorphism $f : \mathbb{C}^4 \to W$ given by $f(e_i) = v_i$ then gives rise to a group isomorphism $f_* : O(4,\mathbb{C}) \to O(W)$. Since $A_{0,0}$ is invariant, we also get a group homomorphism $b : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to O(W)$, which takes $g \times h$ to the linear map that acts as $u \mapsto (g \otimes h)u$. The kernel of b is $\{(e, e), (-e, -e)\}$. The image of b is equal to the image of $SO(4,\mathbb{C})$ under f_*. In fact, the composition $f_*^{-1} \circ b$ gives rise to the group isomorphism

$$\big(SL(2,\mathbb{C}) \times SL(2,\mathbb{C})\big)/\{(e, e), (-e, -e)\} \cong SO(4,\mathbb{C}). \tag{E.5}$$

We already know that $R \in O(W)$. In order to show (6.25) it is therefore enough to prove that R is in the image of $SO(4,\mathbb{C})$ under f_*, i.e. that $\det(R) = 1$. We will show that $\det(R) = -1$ contradicts (E.4) for $s = 2, \bar{s} = 0$.

Let $e_\pm = \left|j = \tfrac{1}{2}, m = \pm\tfrac{1}{2}\right\rangle$ be the standard basis of $V_{1/2}$ and let $e_{\pm\pm} = e_\pm \otimes e_\pm$ be the corresponding basis of W. By $u(1)$-charge conservation, the operator product of $[e_{++} \otimes \varphi_{1,1}](z)$ and $[e_{+-} \otimes \varphi_{1,1}](z)$ must lie in $\mathcal{H}_{[2,0]}$. Therefore, $A_{2,0}(e_{++}, e_{+-}) \neq 0$ or else the operator product would vanish identically (which it does not). On the other hand, again by charge conservation, $A_{2,0}(e_{++}, e_{-+}) = 0$. Define the linear map S via

$$Se_{++} = e_{++}, \qquad Se_{+-} = e_{-+}, \qquad Se_{-+} = e_{+-}, \qquad Se_{--} = e_{--}. \tag{E.6}$$

One can check that $S \in O(W)$ and $\det(S) = -1$. Then $A_{2,0}(Se_{++}, Se_{-+}) = A_{2,0}(e_{++}, e_{+-}) \neq 0$, while $A_{2,0}(e_{++}, e_{-+}) = 0$. It is therefore impossible to satisfy (E.4) for the choice $R = S$. Let now R be an arbitrary element of $O(W)$ with $\det(R) = -1$. Since $O(W)$ has two connected components, the image of $SO(4,\mathbb{C})$ under f_* contains an element x such that $Rx = S$. Then for $u = xe_{++}$ and $v = xe_{-+}$ we have $A_{2,0}(Ru, Rv) = A_{2,0}(Se_{++}, Se_{-+}) \neq 0$, while $A_{2,0}(u, v) = A_{2,0}(xe_{++}, xe_{-+}) = A_{2,0}(e_{++}, e_{-+}) = 0$, so that it is again impossible to satisfy (E.4). Thus the linear map R appearing in (6.25) has necessarily $\det(R) = 1$.

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